EBS 424SW: Vectors and Mechanics

Unit 3: Position vector, Mid-point theorem, Unit vector and 3D vectors

Learning objectives

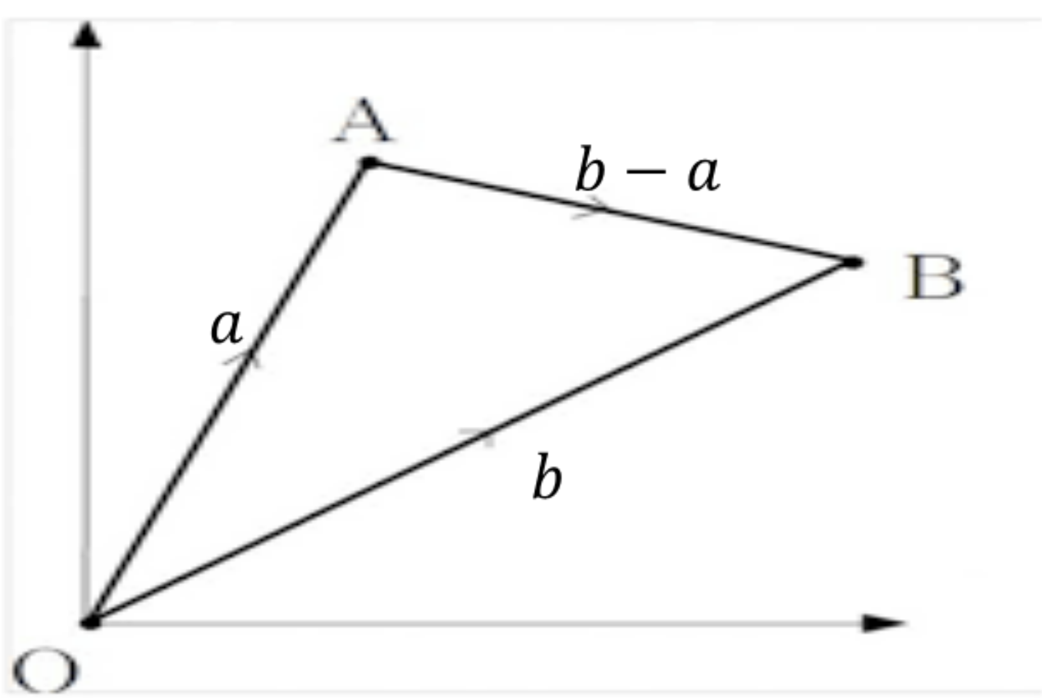
By the end of this unit, you should be able to:

1. express a given vector in terms of position vectors.
2. state and apply mid-point theorem
3. calculate the unit vector of a given vector.
4. describe vectors in 3D.

Position Vector

- A vector that starts from the origin (O) is called a position vector.
- For example, the vector $\overrightarrow{OA}$ which reaches A from the origin is called the position vector of the point A relative the origin, O.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are respectively, the position vectors of the points ***A*** and ***B*** relative to the origin, ***O***, then the position vector of ***B*** relative to ***A*** is given by $\overrightarrow{AB} = b - a$

Proof



$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$
$$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$\overrightarrow{AB} = b - a$$

Position Vector

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are the position vectors of points A and B and M is the mid-point of AB, show that the position vector of M is given by ½ ($\boldsymbol{a} + \boldsymbol{b}$).

Solution

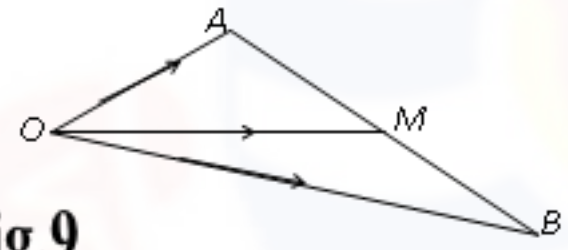


Fig.9

From the diagram; **OM** = **OA + AM**

But **AM = ½AB.** Thus: **OM = OA + ½ AB**

$$= \boldsymbol{OA} + \tfrac{1}{2}(\boldsymbol{OB} - \boldsymbol{OA})$$

$$= \boldsymbol{a} + \tfrac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$

$$= \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) \text{ as required.}$$

This is known as **vector median property of a triangle.**

From **OM = ½ (OA + OB)** we have **OA + OB = 2OM** (*a*) *OAB* is a triangle, where *O* is the origin. Let ***a***, ***b*** be the position vectors of *A* and *B* respectively. If *X* is a point on *AB* such that AX = 2*XB* and *Y* is the mid-point of *OX*, show that $\overrightarrow{BY} = \frac{1}{6}a - \frac{2}{3}b$.

(*b*) What is the resultant of the following five vectors? 3**OA**, 6**BZ**, 2**AO** **AB** and 5**OB**.

Solution

(*a*)

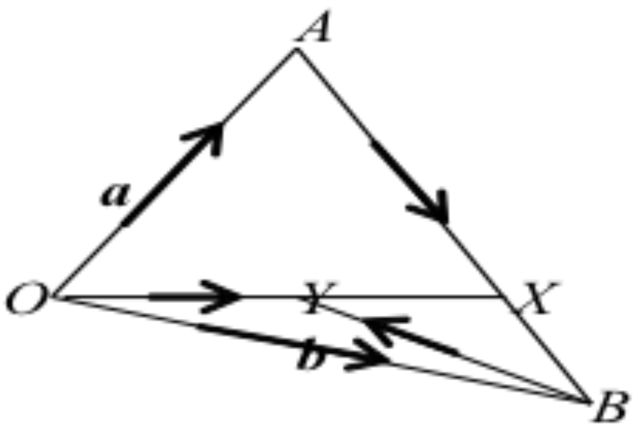


Fig.10

$$\mathbf{BY} = \mathbf{BO} + \mathbf{OY}$$
$$= -\mathbf{OB} + \tfrac{1}{2}\mathbf{OX}$$
$$= -\mathbf{OB} + \tfrac{1}{2}(\mathbf{OA} + \mathbf{AX})$$
$$= -\boldsymbol{b} + \tfrac{1}{2}(\boldsymbol{a} + 2\mathbf{XB})$$

But $\mathbf{AB} = \mathbf{AX} + \mathbf{XB}$

$$= 2\mathbf{XB} + \mathbf{XB}$$
$$= 3\mathbf{XB}$$

Therefore;

$$\mathbf{XB} = \frac{1}{3}\mathbf{AB} = \frac{1}{3}(\mathbf{OB} - \mathbf{OA})$$

Hence $\mathbf{BY} = -\boldsymbol{b} + \tfrac{1}{2}[\boldsymbol{a} + 2(\frac{1}{3})(\mathbf{OB} - \mathbf{OA})] = -\boldsymbol{b} + \tfrac{1}{2}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{a}$

$$= \frac{-6b + 3a + 2(b - a)}{6} = \frac{-6b + 3a + 2b - 2a}{6}$$

$$= \frac{a - 4b}{6} = \frac{1}{6}a - \frac{2}{3}b \text{ as required.}$$

(*c*) $3\mathbf{OA} + 6\mathbf{BZ} + 2\mathbf{AO} + \mathbf{AB} + 5\mathbf{OB} = 3\mathbf{OA} + 6(\mathbf{OZ} - \mathbf{OB}) + 2(-\mathbf{OA}) + \mathbf{OB} - \mathbf{OA} + 5\mathbf{OB}$

$$= 3OA + 6OZ - 6OB - 2OA + OB - OA + 5OB$$

$$= 6OZ$$

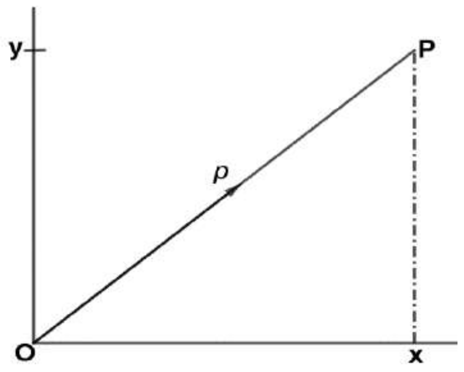
Unit Vector

- A unit vector is defined as a *vector with magnitude one*. In other words, any vector with its length as one, is a unit vector.

- The letters **i** and **j** are used to define unit vectors in the positive directions of the x-and y-axes respectively.

- Any vector can be written in terms of these unit vectors.

Unit Vector



$$|P| = \sqrt{x^2 + y^2}$$

$$\underline{p} = xi + yj$$

In general, if $p = \binom{x}{y}$, then the unit vector in the direction of P is

$$\hat{P} = \frac{p}{|p|}$$

$$\hat{P} = \frac{1}{|p|}\,(xi + yj)$$

$$\hat{P} = \frac{x}{\sqrt{x^2 + y^2}}i + \frac{y}{\sqrt{x^2 + y^2}}j$$

Unit Vector

Example

A, *B* and *C* are the points (1, 2), (4, 3) and (3, –1) respectively. *D* is the mid-point of *BC*.

(i) Write down the position vector of *D*.

(ii) Express **BA** and **CA** in terms of **i** and **j**

(iii) Show that **BA** + **CA** = 2**DA**`

Solution

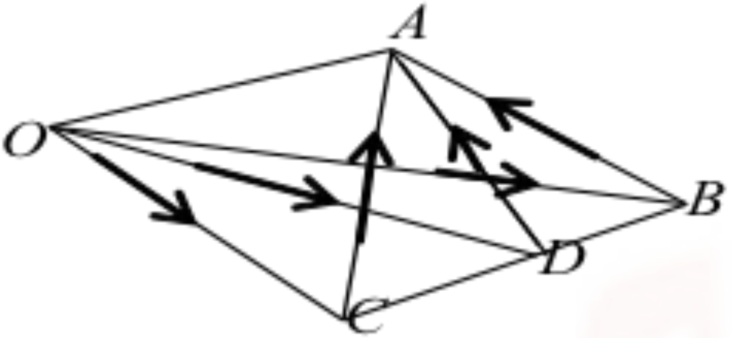


Fig.1

(i) Position vectors of the points are:

$\mathbf{OA} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{OB} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{OC} = 3\mathbf{i} - \mathbf{j}$.

Now $\mathbf{OD} = \mathbf{OB} + \frac{1}{2}\,\mathbf{BC}$

$= \mathbf{OB} + \frac{1}{2}(\mathbf{OC} - \mathbf{OB})$

$= (4\mathbf{i} + 3\mathbf{j}) + \frac{1}{2}[(3\mathbf{i} - \mathbf{j}) - (4\mathbf{i} + 3\mathbf{j})]$

$= 4\mathbf{i} + 3\mathbf{j} + \frac{1}{2}(-\mathbf{i} - 4\mathbf{j})$

$= \frac{7}{2}\mathbf{i} + \mathbf{j}$

(ii) $\mathbf{BA} = \mathbf{OA} - \mathbf{OB} = (\mathbf{i} + 2\mathbf{j}) - (4\mathbf{i} + 3\mathbf{j}) = -3\mathbf{i} - \mathbf{j}$

$\mathbf{CA} = \mathbf{OA} - \mathbf{OC} = (\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} - \mathbf{j}) = -2\mathbf{i} + 3\mathbf{j}$

(iii) $\mathbf{BA} + \mathbf{CA} = (\mathbf{OA} - \mathbf{OB}) + (\mathbf{OA} - \mathbf{OC})$

$= 2\mathbf{OA} - \mathbf{OB} - \mathbf{OC}$

$= 2\mathbf{OA} - (\mathbf{OB} + \mathbf{OC})$

$= 2\mathbf{OA} - 2\mathbf{OD}$ since $\mathbf{OD} = \frac{1}{2}(\mathbf{OB} + \mathbf{OC})$

$= 2(\mathbf{OA} - \mathbf{OD})$

$= 2\mathbf{DA}$ as required.

Unit Vector

If D, E, F are mid-points of the sides BC, CA and AB of triangle ABC, show that
(i) $\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$; (ii) $2\mathbf{AB} + 3\mathbf{BC} + \mathbf{CA} = 2\mathbf{FC}$

Solution

Let the position vectors of points A, B, C, E, F and D be respectively **OA, OB, OC, OE, OF** and **OD** (See Fig.12)

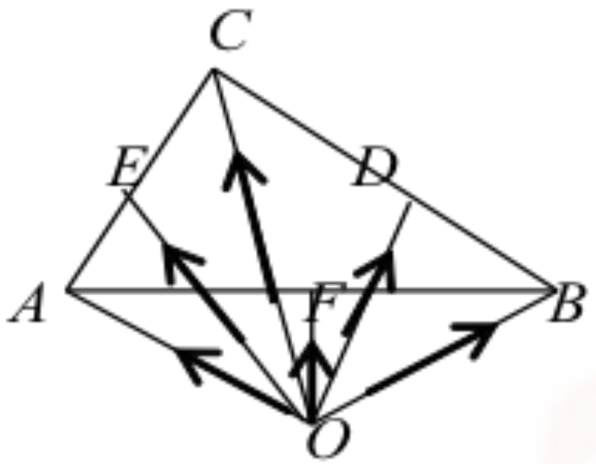


Fig. 2

(i)

$$\begin{aligned}\mathbf{AB} + \mathbf{BC} + \mathbf{CA} &= (\mathbf{OB} - \mathbf{OA}) + (\mathbf{OC} - \mathbf{OB}) \\ &\quad + (\mathbf{OA} - \mathbf{OC}) \\ &= \mathbf{OB} - \mathbf{OA} + \mathbf{OC} - \mathbf{OB} + \mathbf{OA} - \mathbf{OC} \\ &= 0 \text{ as required}\end{aligned}$$

(ii)

$$\begin{aligned}2\mathbf{AB} + 3\mathbf{BC} + \mathbf{CA} &= 2(\mathbf{OB} - \mathbf{OA}) + 3(\mathbf{OC} - \mathbf{OB}) + \mathbf{OA} - \mathbf{OC} \\ &= 2\mathbf{OB} - 2\mathbf{OA} + 3\mathbf{OC} - 3\mathbf{OB} + \mathbf{OA} - \mathbf{OC} \\ &= -\mathbf{OB} - \mathbf{OA} + 2\mathbf{OC} \\ &= -(\mathbf{OB} + \mathbf{OA}) + 2\mathbf{OC} \\ &= -2\mathbf{OF} + 2\mathbf{OC} \qquad [\text{Since } \mathbf{OF} = \tfrac{1}{2}(\mathbf{OA} + \mathbf{OB})] \\ &= 2(\mathbf{OC} - \mathbf{OF}) = 2\mathbf{FC} \text{ as required.}\end{aligned}$$

Unit Vector

A unit vector has been defined as any vector with *magnitude* one. Indeed we can always find a unit vector with the *same direction* as any given vector. A unit vector is often denoted by

$$\hat{a}$$ [called ***a*** cap].

Exercise

Find the unit vector in the direction of $\underline{r} = -10i + 24j$.

$$|r| = \sqrt{(-10)^2 + 24^2} = 26$$

$$\hat{r} = \frac{1}{26}(-10i + 24j)$$

$$\hat{r} = -\frac{5}{13}i + \frac{12}{13}j$$

Exercise

Find the unit vector in the direction of each of the following vectors:

i. $\underline{a} = 6i + 8j$

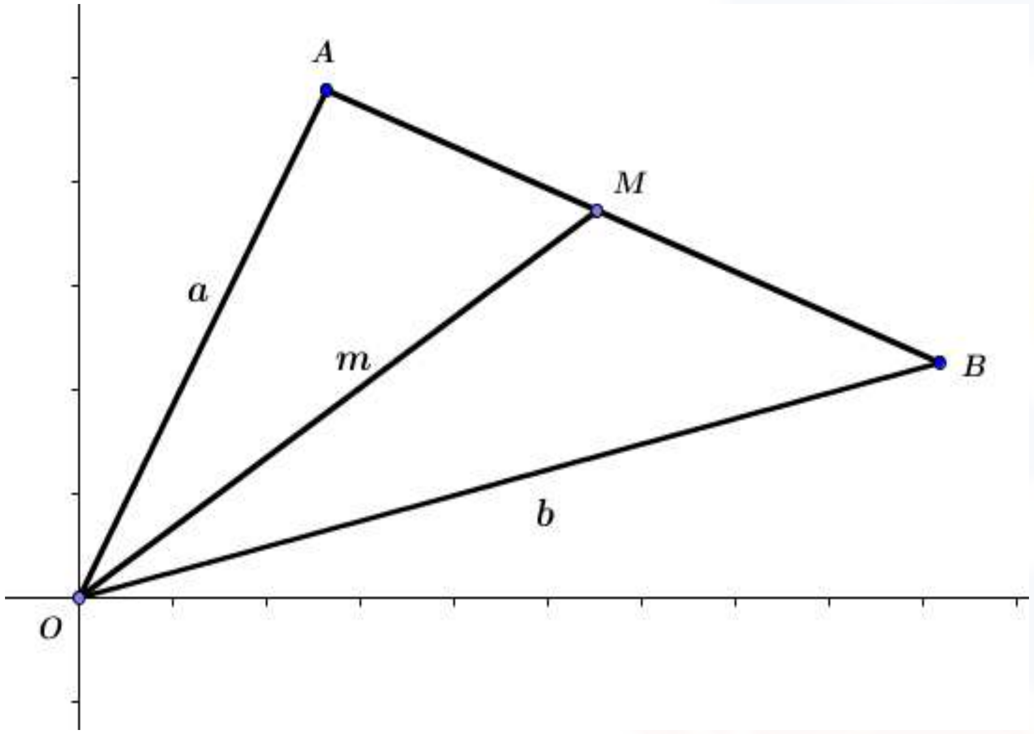
ii. ii. $\underline{r} = -10i + 24j$

iii. iii. $\underline{p} = -\frac{9}{5}i - 8j$

Midpoint Theorem

The midpoint theorem states that if ***a*** and ***b*** are the position vectors ***A*** and ***B*** and ***M*** is the midpoint of ***AB***, the position vector of ***M*** is given by $m = \frac{1}{2}(a+b)$.

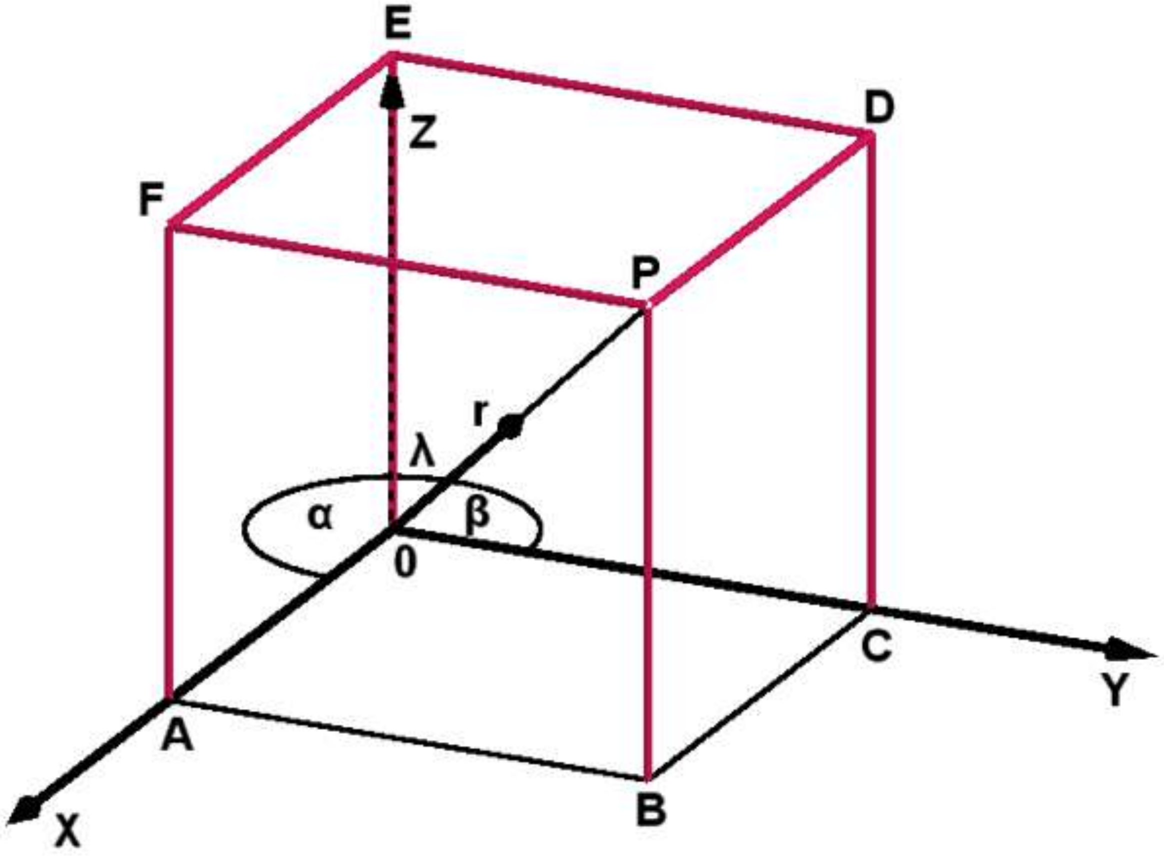
Proof



$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$
$$m = a + \frac{1}{2}(b - a)$$
$$= a - \frac{1}{2}a + \frac{1}{2}b$$
$$= \frac{1}{2}a + \frac{1}{2}b$$
$$m = \frac{1}{2}(a+b).$$

Vectors in space (3-D)

In the 3-D space, we have *x*-axis, *y*-axis and *z*-axis



If $|\overrightarrow{OA}| = x, |\overrightarrow{OC}| = y, and\ |\overrightarrow{OE}| = z.$ Then

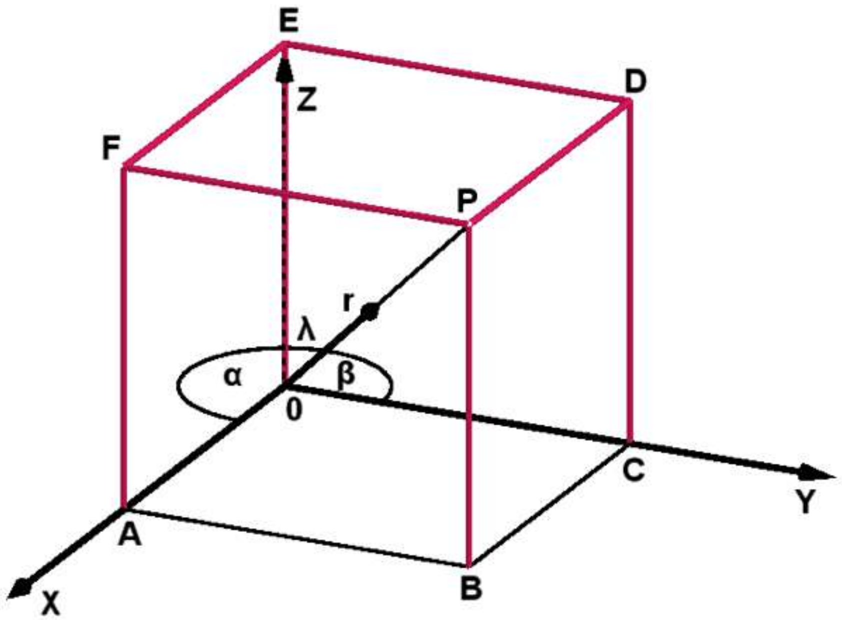
$\overrightarrow{OP} = r = xi + yj + zk$

The length of $\overrightarrow{OP}$ is given by

$$|\overrightarrow{OP}| = |r| = \sqrt{x^2 + y^2 + z^2}$$

DIRECTION COSINES

The direction of a vector in three dimensions is determined by the angles the vectors make with the three axes.



$$\frac{x}{r} = cos\ \alpha \ \Rightarrow x = r\cos \propto$$

$$\frac{y}{r} = \cos\beta \ \Rightarrow y = r\cos\beta$$

$$\frac{z}{r} = \cos\gamma \ \Rightarrow z = r\cos\gamma$$

The numbers $\cos\alpha, cos\ \beta\ and\ cos\ \gamma$ are called the direction cosine of a vector.

Exercise

Find the direction cosines of the vector $r = 3i - 2j + 6k$.

Let $\alpha, \beta\ and, \gamma$ be the angles r makes with x-axis, y-axis and z-axis respectively.

$$r = 3i - 2j + 6k \qquad x = 3, y = -2, z = 6$$

$$r = \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$$

$$\cos\alpha = \frac{x}{r} = \frac{3}{7}$$

$$\cos\beta = \frac{y}{r} = -\frac{2}{7}$$

$$\cos\gamma = \frac{z}{r} = \frac{6}{7}$$

Therefore, the direction cosines of the vector $r = 3i - 2j + 6k$ are $\frac{3}{7}, -\frac{2}{7}\ and\ \frac{6}{7}$

Exercise

Find the direction cosines of each of the following vectors

i. $\underline{r} = 4i + 12j - 5k$

ii. $\underline{p} = i + 2j + 3k$

iii. $\underline{q} = -2i + 4j - k$