Module for B.Ed Primary/Junior High School Programme

2nd Semester
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IoE/MoF/TUC/GHANA CARES TRAINING AND RETRAINING PROGRAMME FOR PRIVATE SCHOOL TEACHERS

FREEDOM AND JUSTICE

Ministry of Finance

TUC

GHANA

Trade Union Congress

VERITAS NOBIS LUMEN

Institute of Education, UCC

EBS 301:CALCULUS

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This course is designed to expose students to the following: Limit of function, Derivation of a function and its interpretation as the rate of change and its application to the determination of a maxima and minima. Differentiation of polynomial functions including the product, quotient and chain rules will be covered. Application of derivatives including linear kinematics, Integration as the inverse (reverse) of difference, Indefinite and definite integrals – Integration of simple continuous functions and Application of integration to areas under curves and kinematics will also be studied. The approaches that would be used in the delivery of this course would prepare trainees to ensure the learning progress of all students by projecting gender roles and issues relating to equity and inclusivity.

Learning outcome(s)

By the end of the unit, the participant will be able to:

- demonstrate a sound knowledge of concepts and procedures in differentiation and integration.
- apply the knowledge acquired in differentiation and integration to solve practical related problems including use of ICT tools.

Unit 1: Limits and Continuity of a Function

Unit Outline

- Session 1: Idea of a Limit
- Session 2: One-Sided Limit
- Session 3: Limits Involving Infinity
- Session 4: Calculating Using Laws of Limits
- Session 5: Intuitive Definition of Continuous Function

Unit 1: Session 1
Idea of a Limit

The concept of a limit or limiting process is essential to the understanding of calculus. In this session we will first illustrate the concept limits by numerically studying the behaviour of a function for values of the independent variable approaching a given number.

Objectives

By the end of this session, you should be able to:

- use a table of values to estimate the limit of a function.
- evaluate limits algebraically.

Consider the behaviour of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives the values of $f(x)$ for values of x close to 2, but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

From the table we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying “the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4”. The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4.$$

In general, we have

$$\lim_{x \to a} f(x) = L$$

To mean that “the limit of $f(x)$, as x approaches a, equals L”.

Alternative notation for

$$\lim_{x \to a} f(x) = L$$

is $f(x) \to L$ as $x \to a$, which is read “$f(x)$ approaches L as x approaches a”.

Remark

In finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matter is, how f is defined when x is near a.

Example 1

Guess the value of

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$$

Using the table below

x	$f(x)$	x	$f(x)$
0.5	0.6667	1.5	0.4000
0.9	0.5263	1.1	0.4762
0.99	0.5025	1.01	0.4975
0.999	0.5003	1.001	0.4998
0.9999	0.5000	1.0001	0.5000

From the table we can guess

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = 0.5.$$

Finding Limits Algebraically

Some of the functions we dealt with in the above examples can be examined algebraically. For instant, $\lim_{x\to 2}(x^2 - x + 2)$, this could be evaluated directly, since $f(x) = x^2 - x + 2$ is a polynomial. That is, $\lim_{x\to 2}(x^2 - x + 2) = 2^2 - 2 + 2 = 4$. Also, $\lim_{x\to 1} \frac{x-1}{x^2-1}$ could be determined as follows

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Example 2

Evaluate $\lim_{x \to 3} f(x)$, if $f(x) = \frac{x^2 - 3x}{x - 3}$.

Solution

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 3x}{x - 3}$$

$$= \lim_{x \to 3} \frac{x(x - 3)}{x - 3}$$

$$= \lim_{x \to 3} x = 3.$$

Example 3

Determine the limits of $f(x)$ as x approaches 2. If $f(x) = \frac{2x^2 - x - 6}{x - 2}$.

Solution

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{2x^2 - x - 6}{x - 2}$$

$$= \lim_{x\to 2} \frac{(x-2)\left(x+\frac{3}{2}\right)}{x-2}$$

$$= \lim_{x\to 2}\left(x+\frac{3}{2}\right)$$

$$= 2 + \frac{3}{2} = \frac{7}{2}.$$

Activity

Guess the value of $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$, using table values.

Exercise 1.1

Evaluate each of the following, if it exists.

1. $\lim_{x\to 1}(x^4 + 3x^2 - 1)$ 2. $\lim_{x\to 1}(2x^3 + 4x - 2)$ 3. $\lim_{x\to 3}\frac{x^2-9}{x-3}$

4. $\lim_{x\to -2}\frac{x^4-2}{2x^2-3x+2}$ 5. $\lim_{x\to 5}\frac{x^2-6x+5}{x-5}$ 6. $\lim_{x\to -3}\frac{x^2+3x}{x^2-x-12}$ 7. $\lim_{x\to 3}\frac{x^3-27}{x-3}$

8. $\lim_{h\to 0}\frac{(-5+h)^2-25}{h}$ 9. $\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$ 10. $\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

Unit 1: Session 2
One-side Limits

In some cases, functions behave differently depending on what side of the function that they are on. In this Session, we consider one-sided limit, which is the behavior of a function on one only one side of the value where the function is undefined or there may be times when we only want to find the limit from one side.

Objectives

By the end of this session, you should be able to:

- evaluate one-sided limits algebraically
- Evaluate one-limits graphically.

One-sided limits are denoted by placing a positive (+) or negative (–) sign as an exponent on the value a.

For example, if we wanted to find a one-sided limit from the left then the limit would look like $\lim_{x \to a^-} f(x)$. This limit would be read as "the limit of $f(x)$ as x approaches a from the left."

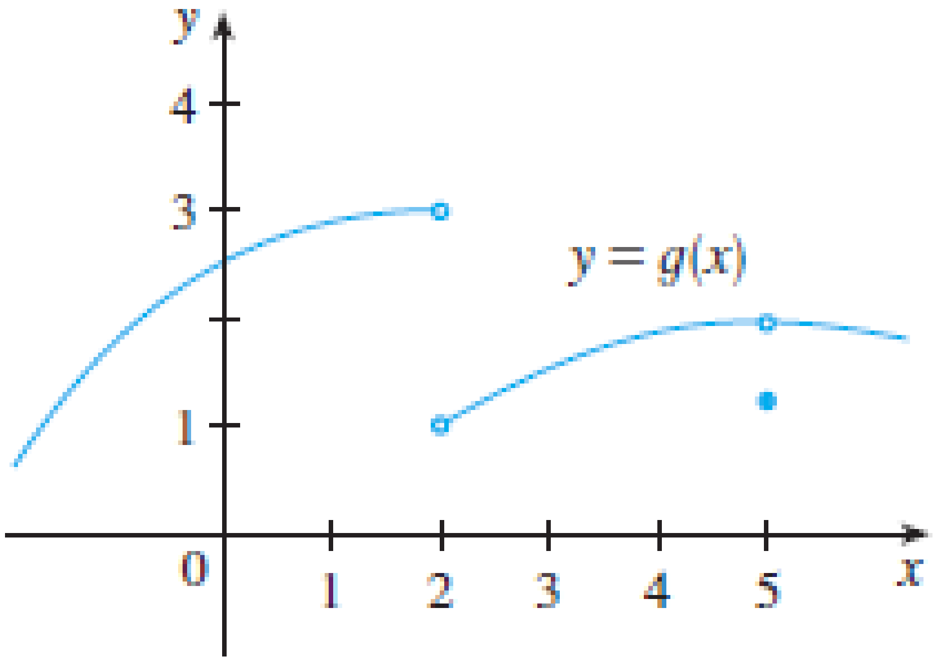
Similarly, A right-handed limit would look like $\lim_{x \to a^+} f(x)$ and would be read as "the limit of $f(x)$ as x approaches a from the right."

Finding one-sided limits are important since they will be used in determining whether the limit of a function exists or not. For the limit of a function to exist both one-sided limits must exist and be equal to the same value.

Thus, $\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = M$ and $L = M$.

Example 1

The graph of the function g is shown below, use it to evaluate the following limits, if it exists.



a. $\lim_{x\to 2^-} g(x)$ b. $\lim_{x\to 2^+} g(x)$ c. $\lim_{x\to 2} g(x)$

d. $\lim_{x\to 5^-} g(x)$ e. $\lim_{x\to 5^+} g(x)$ f. $\lim_{x\to 5} g(x)$

Solution

From the graph we see that the values of g(x) approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

a. $\lim_{x \to 2^-} g(x) = 3$ and b. $\lim_{x \to 2^+} g(x) = 1$.

For c Since the left and right limits are different, we conclude that $\lim_{x \to 2} g(x)$ does not exist.

From the graph we see that the values of g(x) approach 2 as x approaches 5 from the left, but they approach 2 as x approaches 5 from the right. Therefore

d. $\lim_{x \to 5^-} g(x) = 2$ and e. $\lim_{x \to 5^+} g(x) = 2$.

For f. Since the left and right limits are the same, we conclude that $\lim_{x \to 5} g(x) = 2$.

Example 2

Let

$$f(x) = \begin{cases} x^2 + 1 & if\ x < 1 \\ (x-2)^2 & if\ x \geq 1 \end{cases}.$$

Find a. $\lim_{x \to 1^-} f(x)$ b. $\lim_{x \to 1^+} f(x)$ c. $\lim_{x \to 1} f(x)$.

Solution

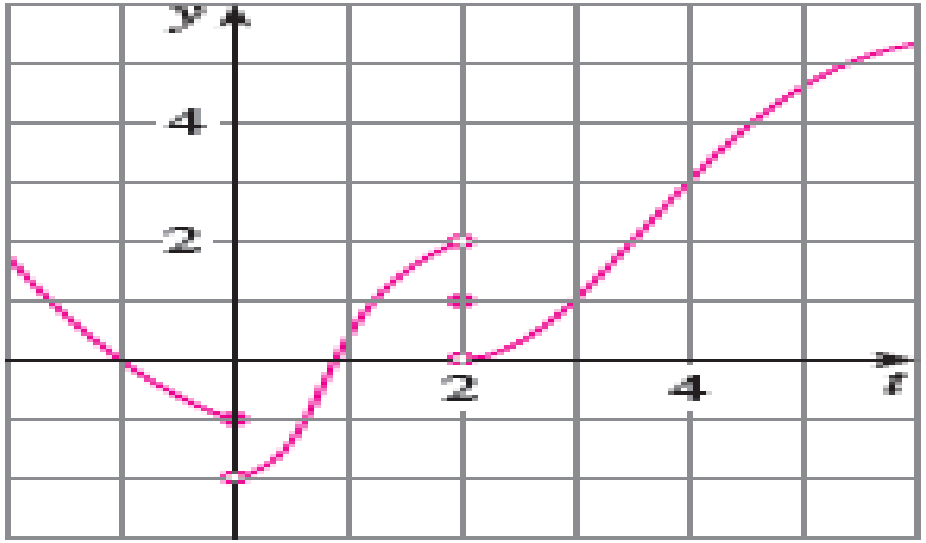
a. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1}(x^2+1) = 2$

b. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1}(x-2)^2 = 1$

c. $\lim_{x \to 1} f(x)$ does not exist.

Exercise 1.2

1. For the function t whose graph is given, state the value of each quantity, if it exists.



a. $\lim_{t \to 0^-} g(t)$ b. $\lim_{t \to 0^+} g(t)$ c. $\lim_{t \to 0} g(t)$

d. $\lim_{t \to 2^-} g(t)$ e. $\lim_{x \to 2^+} g(t)$ f. $\lim_{x \to 2} g(t)$ g. $\lim_{x \to 4} g(t)$

2. Consider the piecewise function

$$f(t) = \begin{cases} 1+t & \text{if } t < -1 \\ t^2 & \text{if} -1 \le t < 1. \\ 2-t & \text{if } t \ge 1 \end{cases}$$

use it to determine each of following quantity, if it exists. If it does not exist, explain why.

a. $\lim_{t \to -1^-} f(t)$,

b. $\lim_{t \to -1^+} f(t)$,

c. $\lim_{t \to -1} f(t)$,

d. $\lim_{t \to 1^-} f(t)$,

e. $\lim_{t \to 1^+} f(t)$,

f. $\lim_{t \to 1} f(t)$,

Unit 1: Session 3
Limits Involving Infinity

We need to know the behavior of $f(x)$ as $x \to \pm\infty$. In this session, we define limits at infinity and show how these limits affect the graph of a function.
We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity.

Objectives

By the end of this session, you should be able to:

- evaluate limits at infinity algebraically

Recall that $\lim_{x \to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For instance, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen numerically in the Table below, as the values of x get larger(to $\pm\infty$), the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim_{x \to \infty} f(x) = 2$.

x	$f(x)$	x	$f(x)$
10	2.1	-10	1.9
100	2.01	-100	1.99
1000	2.001	-1000	1.999
10000	2.0001	-10000	1.9999
100000	2.00001	-100000	1.99999
1000000	2.000001	-1000000	1.999999
10000000	2.0000001	-10000000	1.9999999

Remark

We realized that, as x get larger (approaches $\pm\infty$), $\frac{1}{x}$ get smaller and smaller (approaches zero). Mathematically, $x \to \pm\infty, \frac{1}{x} \to 0$ and $f(x) \to 2$.

Example 1

Evaluate

$$\lim_{x\to\infty} \frac{5x-3}{x^2+2x-1}.$$

Solution

To evaluate this limit, we divide each term (both in the numerator and denominator) by the 'x' with highest power (degree). Thus

$$\lim_{x\to\infty}\frac{5x-3}{x^2+2x-1}=\lim_{x\to\infty}\frac{\frac{5}{x}-\frac{3}{x^2}}{1+\frac{2}{x}-\frac{1}{x^2}}.$$

We notice as $x\to\infty, \frac{2}{x}\to 0, \frac{2}{x}\to 0, \frac{5}{x}\to 0, \frac{1}{x^2}\to 0$ and $\frac{3}{x^2}\to 0$.

Hence

$$\lim_{x\to\infty}\frac{5x-3}{x^2+2x-1}=0.$$

Example 2

Evaluate

$$\lim_{x\to\infty} \frac{1-4x^3}{x^3-2x^2+x+3}.$$

Solution

Dividing every term by x^3, we obtain

$$\lim_{x\to\infty} \frac{1-4x^3}{x^3-2x^2+x+3} = \lim_{x\to\infty} \frac{\frac{1}{x^3}-4}{1-\frac{2}{x}+\frac{1}{x^2}+\frac{3}{x^3}}.$$

Hence

$$\lim_{x\to\infty} \frac{1-4x^3}{x^3-2x^2+x+3} = -4.$$

Exercise 1.3

a. $\lim\limits_{x\to\infty} \frac{x^2-1}{x^2+1}$ b. $\lim\limits_{x\to\infty} \frac{3+5x^2}{x+x^2}$ c. $\lim\limits_{x\to\infty} \frac{3+5x^2}{x+1}$

d. $\lim\limits_{x\to\infty} \frac{x^4}{x^3-2x}$ e. $\lim\limits_{x\to\infty} \sqrt{\frac{1+4x^2}{1+x^2}}$ f. $\lim\limits_{x\to\infty} \frac{3\sqrt{x}}{\sqrt{x}+2}$

Unit 1: Session 4
Calculating Using Laws of Limits

In this session, we shall consider some properties of limits known as the laws of limits. We state and use them without proves.

Objectives

By the end of this session, you should be able to:

- state some of the laws of limits
- use the laws to evaluate limits of functions.

Limit Laws

Suposed that k is a constant and the limits

$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x)$
2. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} [f(x)/g(x)] = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$, if $\lim_{x \to a} g(x) \neq 0$.

These five laws can be stated verbally as follows:

1. The limit of a sum is the sum of the limits.

2. The limit of a difference is the difference of the limits.

3. The limit of a constant times a function is the constant times the limit of the function.

4. The limit of a product is the product of the limits.

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

Note that, $\lim_{x \to a} k = k$.

Example 1

Given that the $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = -3.$

Find

a. $\lim_{x\to 2}[f(x) + 2g(x)]$

b. $\lim_{x\to 2}[f(x)g(x)]$

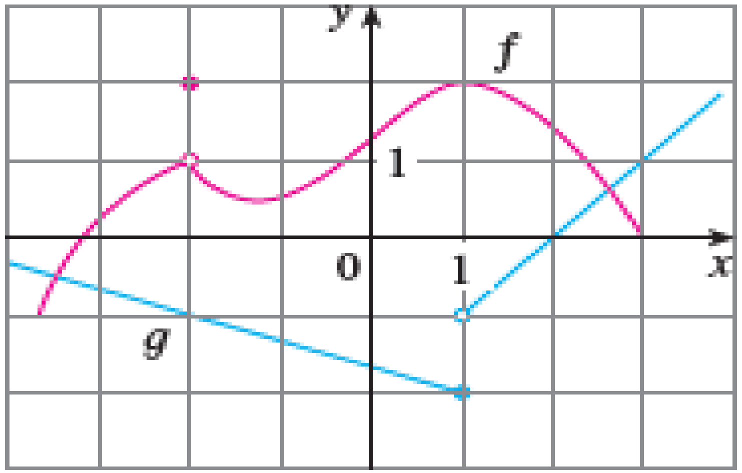
Solution

a. $$\begin{aligned}\lim_{x\to 2}[f(x) + 2g(x)] &= \lim_{x\to 2} f(x) + 2\lim_{x\to 2} g(x)\\ &= 5 + 2(-3)\\ &= 5 - 6 = -1.\end{aligned}$$

b. $\lim_{x \to 2}[f(x)g(x)] = \lim_{x \to 2} f(x) \cdot \lim_{x \to 2}(x) = 5 \times (-3) = -15.$

Example 2

Use the Limit Laws and the graphs of f and g given below to evaluate the
following limits, if they exist.



a. $\lim_{x \to -2} [f(x) + 5g(x)]$ b. $\lim_{x \to 1} [f(x)g(x)]$ c. $\lim_{x \to 2} [f(x)/g(x)]$

Solution

a. From the graph

$$\lim_{x \to -2} f(x) = 1 \text{ and } \lim_{x \to -2} g(x) = -1.$$

Hence

$$\begin{aligned}\lim_{x \to -2} [f(x) + 5g(x)] &= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \\ &= 1 + 5(-1) \\ &= 1 - 5 = -4.\end{aligned}$$

b. From the graph we have $\lim_{x\to 1} f(x) = 2$, but $\lim_{x\to 1} g(x)$ does not exist, because $\lim_{x\to 1^-} g(x) = -2$ and $\lim_{x\to 1^+} g(x) = -1$.

Hence

$\lim_{x\to 1}[f(x)g(x)]$ does not exist.

c. From the graph

$$\lim_{x\to 2} f(x) \approx 1.4 \text{ and} \lim_{x\to 2} g(x) = 0\,.$$

Since the denominator is zero.

$\lim_{x\to 2}[f(x)/g(x)]$ does not exist.

Exercise 1.4

1. Given that

$$\lim_{x\to 2} f(x) = 4, \lim_{x\to 2} g(x) = -2 \text{ and } \lim_{x\to 2} h(x) = 0.$$

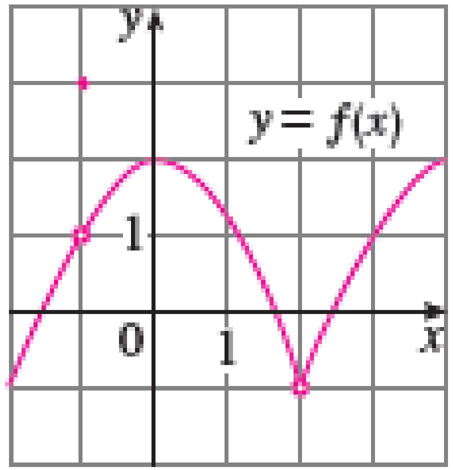
Find the limits that exist. If the limit does not exist, explain why.

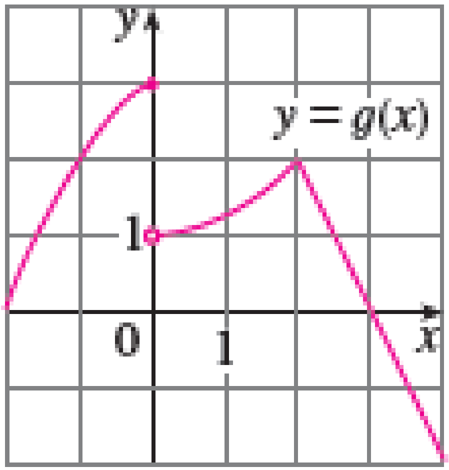
a. $\lim_{x\to -2} [f(x) + 5g(x)]$ b. $\lim_{x\to 2} [g(x)]^3$

c. $\lim_{x\to 2} \frac{3f(x)}{g(x)}$ d. $\lim_{x\to 2} \frac{g(x)}{h(x)}$

e. $\lim_{x\to 2} \frac{g(x)h(x)}{f(x)}$ f. $\lim_{x\to 2} \sqrt{f(x)}$

2. The graphs of f and g are given use them to evaluate each limit, If it exist. If the limit does not exist, explain why.





a. $\lim_{x \to 2} [f(x) + g(x)]$ b. $\lim_{x \to 2} [f(x) - g(x)]$

c. $\lim_{x \to -1} f(x)g(x)$ d. $\lim_{x \to 3} f(x) \backslash g(x)$

e. $\lim_{x \to 2} x^2 f(x)$

f. $f(-1) + \lim_{x \to -1} g(x)$

Exercises 3 – 6, evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 3} (5x^3 - 3x^2 + x - 6)$

4. $\lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim_{t \to -2} \frac{t^4 - 2}{2t^2 - 3t + 2}$

6. $\lim_{x \to 8} (1 + \sqrt[3]{x})(x^3 - 6x^2 + 2)$

Exercises 7-10, evaluate the limit, if it exist.

7. $\lim_{x \to 5} \frac{x^2-6x+5}{x-5}$

8. $\lim_{x \to -3} \frac{x^2+3x}{x^2-x-12}$

9. $\lim_{x \to 5} \frac{x^2-5x+6}{x-5}$

10. $\lim_{x \to 4} \frac{x^2-3x}{x^2-x-12}$

Unit 1: Session 5
Intuitive Definition of Continuous Function

By now, we would have realized that the evaluation of the limit of a function as *x* approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called continuous at a. In this session we will consider the mathematical definition of continuity of function which is very close with the meaning of the word continuity in everyday language.

Objectives

By the end of this session, you should be able to:

- Define continuity at a point
- Determine the continuity of a function at a point.

For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.

That is, if a function f is continuous at a point a, mathematically

$f(a)$ is defined

$\lim_{x \to a} f(x)$ exists

$$\lim_{x \to a} f(x) = f(a).$$

Example 1

Where are each of the following functions discontinuous?

a. $f(x) = \frac{x^2-x-2}{x-2}$ b. $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

c. $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$.

Solution

a. Notice that, $f(x)$ is not defined at $x = 2$, so the function is not continuous at $x = 2$.

b. We see that $f(0) = 1$, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2}$ does not exist, hence, the function is not continuous at $x = 0$.

c. We see that $f(2) = 1$ and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2}$$

$$= \lim_{x\to 2}(x+1) = 3.$$

But $\lim_{x\to 2} f(x) = 3 \neq f(2) = 1$.

Hence, the function is not continuous at $x = 2$.

Theorem

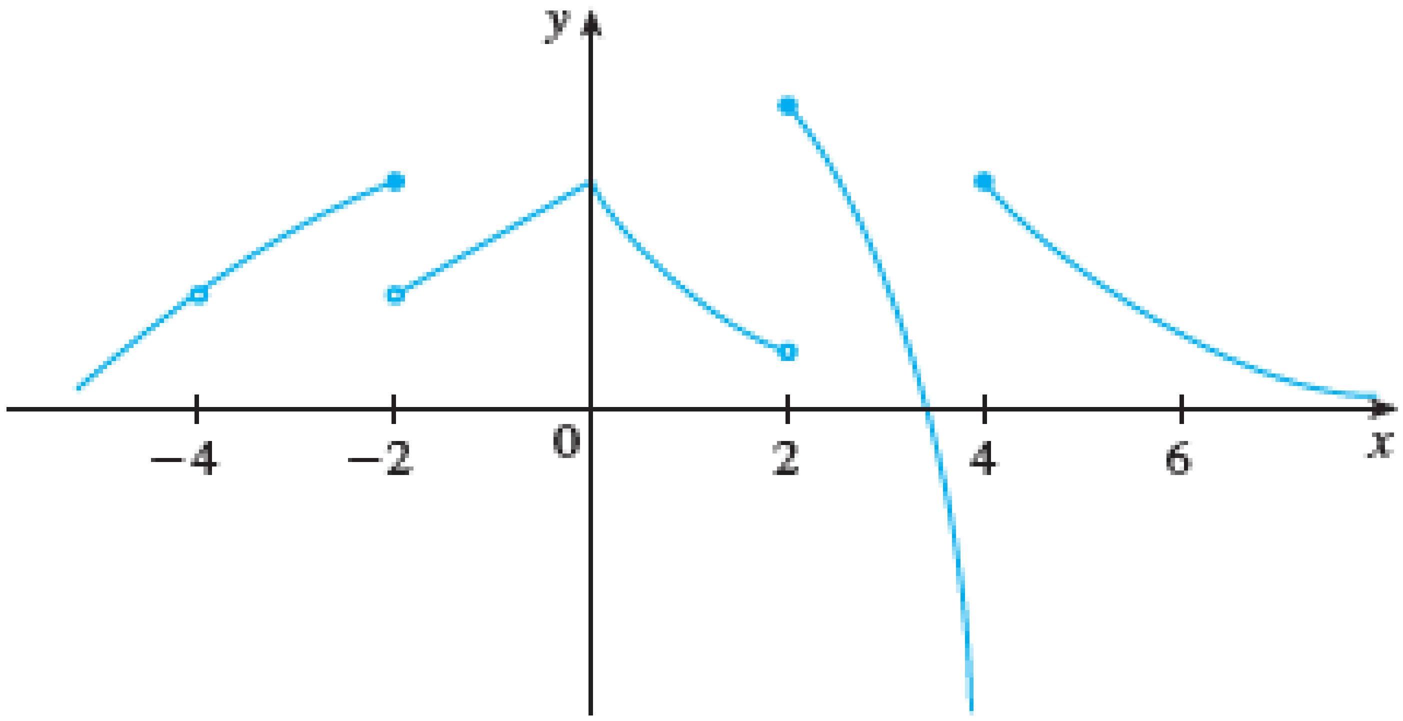
If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

a. $f(x) + g(x)$ b. $f(x) - g(x)$ c. $cf(x)$

e. $f(x).\, g(x)$ f. $\frac{f(x)}{g(x)}$, if $g(a) \neq 0$.

Example 2

From the graph of f , state the numbers at which f is discontinuous and explain why.



Solution

They are: $-2, \;\; 2$ and 4

Consider $x = -2$

We have $\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$, hence the limit does not exist at $x = -2$.

Consider $x = 2$

We have $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, hence the limit does not exist at $x = 2$.

Consider $x = 4$

We have $\lim_{x \to 4^-} f(x) = \infty$, hence the limit does not exist at $x = 4$.

Exercise 1.2

Exercises 1 – 4. Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

1. $f(x) = (x + 2x^2)^4, a = -1.$

2. $g(t) = \frac{t^2+5t}{2t+1},\ a = 2$

3. $f(x) = 2\sqrt{3x^2 + 1},\ a = 1$

4. $f(x) = 3x^4 - 5x + \sqrt[3]{x^2 + 4},\ a = 2.$

Exercise 5 – 8. Explain why the function is discontinuous at the given number a.

5. $f(x) = \frac{1}{x+2}$, $a = -2$.

6. $f(x) = \begin{cases} \frac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$, $a = -2$

7. $f(x) = \begin{cases} x+3 & \text{if } \leq -1 \\ 2^x & \text{if } x > 1 \end{cases}$, $a = -1$

8. $f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$, $a = 1$

Unit 2: Differentiation

Unit Outline

Session 1: Differentiation from First Principles
Session 2: Differentiation of Polynomials
Session 3: Rate of Change
Session 4: Tangents and Normals
Session 5: Product and Quotient Rules
Session 6: The Chain Rule

Unit 2: Session 1
Differentiation from First Principles

In this session we introduce the idea of derivative which is a method of differentiation from first principle and it gives what we call the gradient function. In practice, working with it is very tedious, as such its value is in establishing a formal basis for differentiation rather than as a working tool.

Objectives

By the end of this session, you should be able to:

- obtain gradient function from first principle.
- evaluate the gradient of a function at a given point.

The derivative of a function f is denoted as f' and it is so called because it is derived from the function f. If we have $y = f(x)$, then the derivative is denoted as $\frac{dy}{dx}$.

The derivative f' is also a function (i.e., gradient function) whose value at x is given by $f'(x)$. Geometrically, f' is interpreted as the slope of the tangent line to the graph of f at the point $(x,\ f(x))$.

By definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Example 1

If $f(x) = x^3 - x$, use the definition given to find $f'(x)$.

Solution

We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1)$$

$$= 3x^2 - 1$$

Example 2

Given that $f(x) = \frac{1-x}{2+x}$, use the definition of derivative to determine $f'(x)$.

Solution

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h}$$

$$= \lim_{h \to 0} \frac{(2-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3h}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2 + x + h)(2 + x)}$$

$$= \frac{-3}{(2 + x)^2}$$

Exercise 2.1

Find the derivative of each of the following using the definition (first principle).

1. $f(x) = 2x^2 + 5x$
2. $f(x) = (x + 3)^2$
3. $f(x) = 2 - x^2$
4. $f(x) = x^3 - 3x + 2$

Unit 2: Session 2

Differentiation of Polynomials

In this session we learn how to differentiate constant functions, power functions and polynomials functions. We will state and use the Power Rule to differentiate polynomials and other power functions without proof.

Objectives

By the end of this session, you should be able to:

- State the power rule.
- Use the power rule to differentiate polynomials and other power functions.

Polynomials are one of the simplest functions to differentiate. When taking derivatives of polynomials, we primarily make use of the power rule.

If n is a real number, then the derivative of $y = x^n$ is given as

$$\frac{dy}{dx} = nx^{n-1} \quad (Power\ Rule).$$

Example 1

Use Power Rule to determine the derivatives of the following functions:

a. $y = x^2$ b. $y = x^7$ c. $y = x^{-4}$ d. $y = \sqrt{x}$

Solution

a. $\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x^{2-1} = 2x$

b. $\frac{dy}{dx} = \frac{d}{dx}(x^7) = 7x^{7-1} = 7x^6$

c. $\frac{dy}{dx} = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}$

d. $\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Note that the derivative of a constant is zero. Thus

$$\frac{d}{dx}(k) = 0$$

The Constant Multiple Rule. If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}(cf) = c\frac{d}{dx}(f).$$

Example 2

Determine the gradient functions of the following polynomials

a. $f(x) = 5x^3$ b. $f(x) = -3x^4$

Solution

a. $f'(x) = 5 \times 3x^{3-1} = 15x^2$ b. $f'(x) = -3 \times 4x^{4-1} = -12x^3$

The Sum and Difference Rules. If f and g both differentiable function, then

$$\frac{d}{dx}(f \pm g) = \frac{d}{dx}(f) \pm \frac{d}{dx}(g).$$

Example 2

Determine the gradient of the given functions at the indicated point.

a. $f(x) = 5x^3 + 2x^5,\ x = 1.$ b. $f(x) = 2 + x - 3x^4, \quad x = -2.$

Solution

a. $f'(x) = 15x^2 + 10x^4, \Rightarrow f'(1) = 15(1)^2 + 10(1)^4 = 15 + 10 = 25$

b. $f'(x) = 0 + 1 - 12x^3, \ \Rightarrow f'(-2) = 1 - 12(-2)^3 = 1 + 96 = 97$

Exercise 2.2

Exercises 1-7, evaluate, differentiate the functions

1. $f(x) = 6.1x^2 + 2.3$ 2. $f(x) = 2x^3 - 3x^2 - 4x$

3. $f(t) = t^2(1 - 2t)$ 4. $g(x) = \frac{7}{4}x^2 - 3x + 12$

5. $f(t) = 1.4t^5 - 2.5t^2 + 6.7$ 6. $g(t) = 2t^{-\frac{3}{4}}$

7. $y = x^{\frac{5}{3}} - x^{\frac{2}{3}}$

Unit 2: Session 3
Rate of Change

In this session the derivative of a function will be seen to be interpreted as a rate of change in any of the natural or social sciences or engineering. For instance, a biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences.

Objectives

By the end of this session, you should be able to:

- solve practical problems involving rate of change.

Suppose *y* is a quantity that depends on another quantity *x*. Thus *y* is a function of *x* and we write $y = f(x)$. If *x* changes from x_1 to x_2, then the change in *x* (also called the **increment** of *x*) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in *y* is

$$\Delta y = f(x_2) - f(x_1).$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1,\ x_2]$.

The limit of these average rates of change is called the **(instantaneous) rate of change of *y* with respect to *x*** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at the point $(x_1, f(x_1))$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_2} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

We recognize this limit as being the derivative $f'(x_1)$.

Example 1

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

a. What is the meaning of the derivative $f'(x)$? What are its units?

b. In practical terms, what does it mean to say that $f'(1000) = 9$?

Solution

a. The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced (Economists call this rate of change the *marginal cost*).

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C / \Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

b. The statement that $f(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is $9 per yard. (When $x = 1000$, C is increasing 9 times as fast as x.)

Example 2

Water is flowing out of bath and the depth of the water, y cm, at time t seconds is given by

$$y = 16 - \frac{1}{8}t - \frac{1}{4}t^3.$$

Find the rate at which the depth of water is decreasing at 2 seconds and at 3 seconds.

Solution

The rate of change of water level is given by

$$\frac{dy}{dt} = -\frac{1}{8} - \frac{3}{4}t^2.$$

When $t = 2, \frac{dy}{dt} = -\frac{1}{8} - \frac{3}{4}(2)^2 = -3\frac{1}{8}$.

When $t = 3, \frac{dy}{dt} = -\frac{1}{8} - \frac{3}{4}(3)^2 = -6\frac{7}{8}$.

Since in both cases $\frac{dy}{dt} < 0$, it means that after 2 seconds and 3 seconds, the depth of water is decreasing at $3\frac{1}{8}$ cm/s and $6\frac{7}{8}$ cm/s respectively

Example 3

The radius of a spherical balloon is increasing at 0.5 cm/s. at the instant when the radius is 4 cm, find the rate at which

a. the surface area is increasing.

b. the volume is increasing.

Solution

Let the radius the balloon be r cm, surface area A cm^2, and the volume V cm^2.

a. Given $\frac{dr}{dt} = 0.5$, we have to find $\frac{dA}{dt}$. Now $A = 4\pi r^2$, $\Rightarrow \frac{dA}{dr} = 8\pi r$.

But $\quad \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 8\pi r \times 0.5 = 4\pi r.$

Hence, at the instant when $r = 4, \frac{dA}{dt} = 4(4)\pi = 16\pi.$

Therefore, the surface area is increasing at the rate of 16π cm^2/s.

b. we have to find $\frac{dV}{dt}$. Now $V = \frac{4}{3}\pi r^3, \Rightarrow \frac{dV}{dr} = 4\pi r^2$.

But $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \times 0.5 = 2\pi r^2$.

Hence, at the instant when $r = 4$, $\frac{dV}{dt} = 2\pi(4)^2 = 32\pi$.

Therefore, the volume is increasing at the rate of 32π cm^3/s.

Exercise 2.3

1. The cost (in cedis) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

a. Find the average rate of change of C with respect to x when the production level is changed

i. From $x = 100$ to $x = 105$.

ii. From $x = 100$ to $x = 101$.

b. Find the instantaneous rate of change of C with respect to x when $x = 100$.

2. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100{,}000\left(1 - \frac{1}{60}t\right)^2, 0 \le t \le 60.$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0$, 10, 20, 30, 40, 50 and 60 minutes, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest?

3. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.

a. What is the meaning of the derivative $f'(x)$? What are its units?

b. What does the statement $f'(800) = 17$?

c. Do you think the value of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

4. The side of a cube is increasing at rate of 6 cm/s. find the rate of increase of volume when the length of a side is 9 cm.

5. The area of surface of a sphere is $4\pi r^2$, r being the radius. Find the rate of change of the area in square cm per second when $r = 2$ cm, given that the radius increases at the rate of 1 cm/s.

6. The volume of a cube is increasing at the rate of 2 cm^3/s. find the rate of change of the side of the base when its length is 3 cm.

Unit 2: Session 4
Tangents and Normals to a Curve

In this session we consider two straight lines known as tangent and normal to a given a curve at a point. We defined and determine their equations.

Objectives

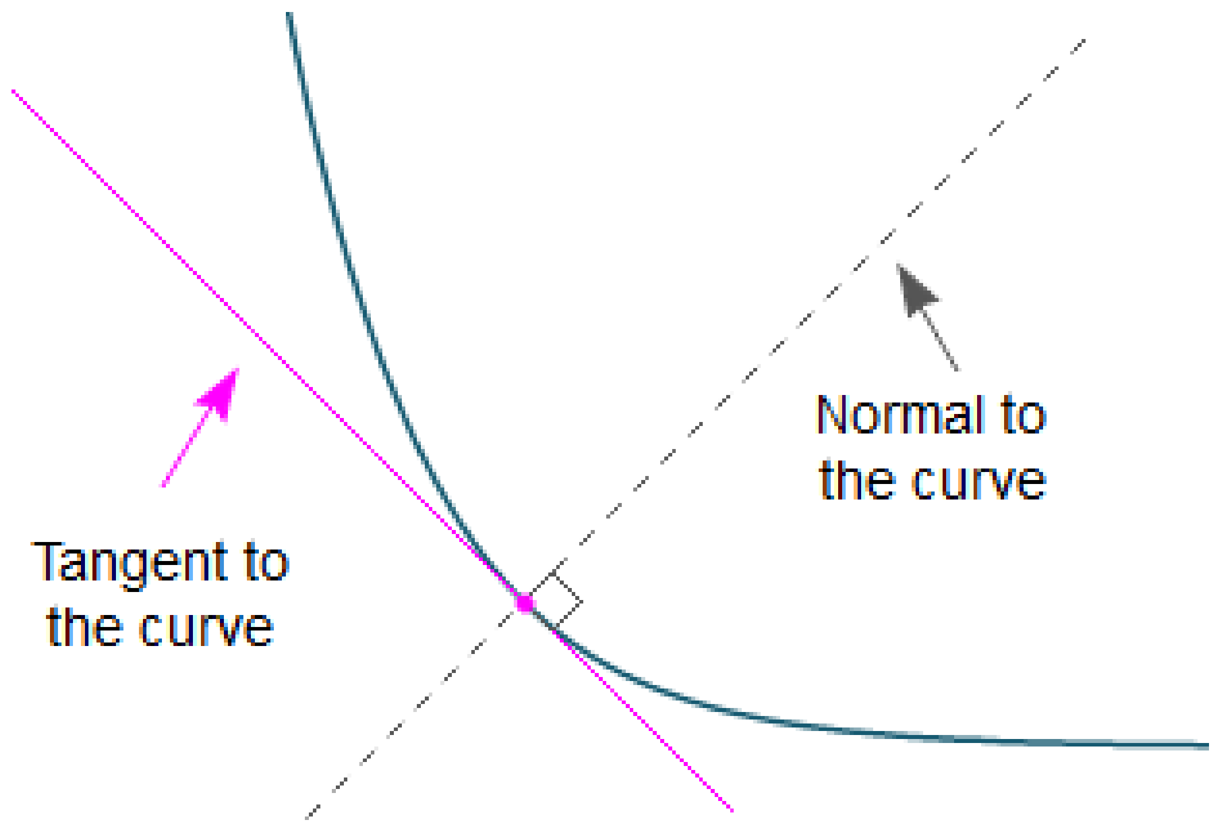
By the end of this session, you should be able to:

- determine the equation of tangent to a curve at a given point.
- determine the equation of normal to a curve at a given point.

We noted that the derivative of a function $y = f(x)$, denoted by $\frac{dy}{dx}$ $or\ f'(x)$, is the gradient (function) of the tangent to the curve $f(x)$ at any point with coordinates $(x,\ y)$ on the curve. Thus, at a given point $P(x_1,\ y_2)$ on the curve $y = f(x)$, the gradient of the tangent at P is the value of $f'(x)$ evaluated a P, namely $f'(x_1)$. Hence, we can write the equation of the tangent and normal to the curve of $f(x)$ at $P(x_1,\ y_2)$.

Definition

A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point. (See the figure in the next slide)



Graph showing the tangent and the normal to a curve at a point.

Example 1

Find the equation the tangent to the curve $y = x^3$ at the point $(2,\ 8)$.

Solution

We have $y = x^3$ and the gradient function is $f'(x) = 3x^2$.

When $x = 2$ the gradient of the tangent at $(2,\ 8)$ is $f'(2) = 3(2)^2 = 12$.

Then given any arbitrary point $(x,\ y)$ on the tangent line, the equation of tangent line is given as

$$\frac{y-8}{x-2} = 12, \Rightarrow y - 8 = 12(x-2).$$

Hence, the equation of the tangent is $y = 12x - 16$.

Example 1

Find the equations of the tangent and the normal to the curve $y = 2 - x - x^2$ at the point where $x = 2$.

Solution

Given $y = 2 - x - x^2$ and $y = f(x)$ the gradient function is $f'(x) = -1 - 2x$.

At $x = 2$, the gradient is $f'(2) = -1 - 2(2) = -5$.

When $x = 2, y = 2 - (2) - (2)^2 = -4$.

Thus, the gradient of the tangent to the curve at $P(2, -4)$ is -5.

Hence, given any arbitrary point on the tangent line (x, y) the equation of tangent is given as

$$\frac{y+4}{x-2} = -5, \Rightarrow y+4 = -5(x-2).$$

Hence, the equation of the tangent is $y = -5x + 6$.

The normal to the curve at $P(2, -4)$ is perpendicular to the tangent at P and its gradient is the ***negative reciprocal*** of the gradient of the tangent.

Hence, the gradient of the normal is $-\frac{1}{f'(2)} = -\left(\frac{1}{-5}\right) = \frac{1}{5}$.

Therefore, the equation of the normal at $P(2, -4)$ is given by

$$\frac{y+4}{x-2} = \frac{1}{5}, \qquad \Rightarrow 5(y+4) = x-2.$$

And the equation is $y = \frac{1}{5}x - \frac{22}{5}$.

Exercise 2.4

1. Find the equations of the tangents and normal to the following curves at the points corresponding to the given values of x.

a. $y = x^2,\ x = 2;$

b. $y = 3x^2 + 2,\ x = 4;$

c. $y = 3x^2 - x + 1,\ x = 0;$

d. $y = 3 - 4x - 2x^2,\ x = 1;$

e. $y = 9x - x^3,\ x = -3.$

2. Find the equation of the tangent to the curve $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y −axis.

3. Find the values of x for which the gradient function of the curve $y = 2x^3 + 3x^2 - 12x + 3$ is zero. Hence, find the equations of the tangents to the curve which are parallel to the x −axis.

Unit 2: Session 5
Product and Quotient Rules

In this session we state and use product and quotient rules for finding derivatives without having to use the definition directly. These differentiation rules enables us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential functions, logarithmic functions, and trigonometric functions.

Objectives

By the end of this session, you should be able to:

- Differentiate functions using product rule.
- Differentiate functions using quotient rule.

The Product Rule

Consider $h(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are both differentiable functions, the derivative of $h(x)$is given by

$$h'(x) = f(x)g'(x) + g(x)f'(x).$$

In words, the Product Rule says that the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Example 1

Find the derivatives of the following functions

a. $h(x) = (2x + 3)(x^3 + 2x)$

b. $h(x) = (3x^4 - 2)(1 - x^2)$.

Solution

a. Let $f(x) = 2x + 3$ and $g(x) = x^3 + 2x$.

We have

$$f'(x) = 2 \text{ and } g'(x) = 3x^2 + 2.$$

It means

$$h'(x) = (2x + 3)(3x^2 + 2) + (x^3 + 2x)(2)$$

$$= 6x^3 + 4x + 9x^2 + 6 + 2x^3 + 4x$$

$$= 8x^3 + 9x^2 + 8x + 6.$$

b. Let $f(x) = 3x^4 - 2$ and $g(x) = 1 - x^2$.

We have

$$f'(x) = 12x^3 \text{ and } g'(x) = -2x.$$

It means

$$h'(x) = (3x^4 - 2)(-2x) + (1 - x^2)(12x^3)$$
$$= -6x^5 + 4x + 12x^3 - 12x^5$$
$$= 4x + 12x^3 - 18x^5.$$

Example 2

If $f(x) = \sqrt{x}g(x)$, where $g(4) = 2$ and $g'(4) = 3$. Find $f'(4)$.

Solution

Let $h(x) = \sqrt{x}, \Rightarrow h'(x) = \frac{1}{2\sqrt{x}}$.

We have

$$f'(x) = h'(x)g(x) + g'(x)h(x)$$
$$= \frac{1}{2\sqrt{x}}g(x) + g'(x)\sqrt{x}$$

Given $g(4) = 2$ and $g'(4) = 3$, we obtain

$$f'(4) = \frac{1}{2\sqrt{4}}g(4) + g'(4)\sqrt{4}$$
$$= \frac{1}{4} \times 2 + 3 \times 2$$
$$= \frac{1}{2} + 6 = 6.5$$

The Quotient Rule

Consider

$$h(x) = \frac{f(x)}{g(x)}, \text{for } g(x) \neq 0$$

where $f(x)$ and $g(x)$ are both differentiable functions, the derivative of $h(x)$is given by

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

In words, the Quotient Rule says that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Example 3

Differentiate the following with respect to *x*.

a. $\frac{x-1}{x+1}$ b. $\frac{x^2}{2x+1}$

Solution

a. Let $h(x) = \frac{x-1}{x+1}, f(x) = x - 1$ and $g(x) = x + 1$.

We have

$$f'(x) = 1 \text{ and } g'(x) = 1.$$

It means

$$h'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$$

$$= \frac{x + 1 - x + 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

b. Let $h(x) = \frac{x^2}{2x+1}, f(x) = x^2$ and $g(x) = 2x + 1$.

We have

$$f'(x) = 2x \text{ and } g'(x) = 2.$$

It means

$$h'(x) = \frac{(2x+1)(2x) - (x^2)(2)}{(2x+1)^2}$$

$$= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2} = \frac{2x^2 + 4x}{(2x+1)^2}$$

Exercise 2.5

Exercises 1 to 4, differentiate the functions with respect to x

1. $f(x) = (2x+3)(x^2-7x)$

2. $f(x) = (1-x^3)(3x^2-x)$

3. $f(x) = \frac{3x^2-x}{5x+2}$

4. $f(x) = \frac{\sqrt{x}}{1+x}$

5. Suppose that, $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$ and $g'(4) = -3$.
Find $h'(4)$ if

a. $h(x) = 3f(x) + 8g(x)$

b. $h(x) = f(x)g(x)$

c. $h(x) = \frac{f(x)}{g(x)}$

d. $h(x) = \frac{g(x)}{f(x)+g(x)}$

Unit 2: Session 6
The Chain Rule

In this session we state and use the chain rule for finding derivatives without having to use the definition directly. These differentiation rules enables us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential functions, logarithmic functions, and trigonometric functions.

Objectives

By the end of this session, you should be able to:

- Differentiate functions using the chain rule rule.

The Chain rule

If we can write $y = f(g(x))$, that is, $h(x) = f(g(x))$. We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $h(x) = f(g(x))$ in terms of the derivatives of ff and g. It turns out that the derivative of the composite function $h(x)$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule*.

Therefore, if g is differentiable at x and f is differentiable at $g(x)$, the composite function $h(x) = f(g(x))$ is differentiable at x and $h'(x)$ is given by

$$h'(x) = f'\big(g(x)\big) \cdot g'(x)$$

or if we let $y = f(u)$and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 1

Differentiate the following with respect to x.

a. $(3x+1)^4$

b. $(2x^2+1)^{\frac{1}{2}}$.

Solution

Let $h(x) = (3x+1)^4$.

$$\Rightarrow h'(x) = 4(3x+1)^3 \cdot (3x+1)'$$

$$= 4(3x+1)^3 \cdot (3)$$

$$= 12(3x+1)^3.$$

b. Let $h(x) = (2x^2 + 1)^{\frac{1}{2}}$.

$$\Rightarrow h'(x) = \frac{1}{2}(2x^2 + 1)^{-\frac{1}{2}} \cdot (2x^2 + 1)'$$

$$= \frac{1}{2}(2x^2 + 1)^{-\frac{1}{2}} \cdot (4x)$$

$$= \frac{2x}{\sqrt{2x^2 + 1}}.$$

Example 2

Differentiate $f(x) = (1 - x^2)^{-2}$ with respect x.

Solution

Let $y = (1 - x^2)^{-2}$ and $u = 1 - x^2$, $\quad \Rightarrow \frac{du}{dx} = -2x$.

$\Rightarrow y = u^{-2}$ and $\frac{dy}{du} = -2u^{-3}$.

$$\therefore \ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -2u^{-3} \cdot -2x$$

$$= 4x(1 - x^{-2})^{-3}$$

Example 3

If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$. Find $h'(1)$.

Solution

Given $h(x) = \sqrt{4 + 3f(x)}$

$$\Rightarrow h'(x) = \frac{1}{2}\left(4 + 3f(x)\right)^{-\frac{1}{2}} \cdot 3f'(x).$$

Hence

$$\begin{aligned} h'(1) &= \frac{1}{2}\left(4 + 3f(1)\right)^{-\frac{1}{2}} \cdot 3f'(1) \\ &= \frac{1}{2}(4 + 3 \times 7)^{-\frac{1}{2}} \cdot 3 \times 4 \\ &= \frac{1}{2}(25)^{-\frac{1}{2}} \cdot 12 = \frac{6}{5}. \end{aligned}$$

Exercise 2.6

Exercises 1 to 4, differentiate each function using chain rule.

1. $f(x) = (4x + 5)^3$ 2. $f(x) = (6x - 5)^{-\frac{1}{2}}$

3. $f(t) = (4t + 1)^5$ 4. $f(t) = (7t - 3)^{\frac{3}{2}}$

Exercises 5 to 8, write the composite function in the form $f(g(x))$.
[Identify the inner function $u = g(x)$ and the outer function $y = f(u)$]
Then find the derivative $\frac{dy}{dx}$.

5. $y = \sqrt[3]{1 + 4x}$

6. $y = (2x^3 + 5)^4$

7. $y = (5x^6 + 2x^3)^4$

8. $y = \frac{1}{\sqrt[3]{x^2 - 1}}$