# Module for B.Ed Primary/Junior High School Programme

**2nd Semester**
**April, 2023**

**IoE/MoF/TUC/GHANA CARES TRAINING AND RETRAINING PROGRAMME FOR PRIVATE SCHOOL TEACHERS**

**Ministry of Finance** **Trade Union Congress** **Institute of Education, UCC**

# MATHEMATICAL INVESTIGATIONS
# EBS 417

# COURSE LEARNING OUTCOME

By the end of the course, the student will be able to:

- Demonstrate a sound knowledge of the topics and apply them in real life situations

- Apply the knowledge of various problem solving models acquired in the course to solve real life problems

- Pose and create relevant mathematics tasks and solve them using appropriate heuristics and tools (including ICT tools).

# INDICATOR

- Explain what mathematical investigation is and why engage in investigations in mathematics at the basic school level

- Identify and use the problem solving models suggested by G. Polya and J. Mason to solve real life mathematical problems/puzzles

- Do some investigations involving numbers and number patterns, algebra, geometry, logic, and use of straight lines to make patterns.

# UNIT 1

# PROBLEM, PROBLEM SOLVING AND INVESTIGATIONS IN

# MATHEMATICS

## UNIT OUTLINE

Definition of Problem in Mathematics

Definition of Problem Solving in Mathematics

Definition of Mathematical Investigation

Problem Solving and Investigation compared

Values of Teaching through Problem solving

Values of Teaching through Mathematical Investigations

# Definition of Problem in Mathematics

- The word problem is coined from the Greek word “problema”, which means “something thrown forward”.
- A mathematics problem is a perplexing question or situation.
- It does not offer an immediate solution. It involves some aspect of mathematics which requires thinking at a level beyond memorization.

- A problem usually refers to non-routine tasks for which students do not have an immediate method of solution and those tasks that arise commonly in everyday life for which there may be several methods of solution (Sheffield and Cruikshank, 2000).
- The students know what is asked, but usually do not know a direct way of doing it.
- The goals are clearly defined but the methods or procedures for solving are not readily apparent.

- Problems can range from well-structured to ill-structured, depending on how clear-cut the goal is and how much structure is provided for solving the problem.
- A mathematical problem is a task that is amenable to being represented, analyzed, and possibly solved, with the methods of mathematics.
- This can be a real-world situation, such as computing the orbits of the planets in the solar system, or a task of a more abstract nature.

- Problem solving is usually defined as formulating new answers, going beyond the simple application of previously learned rules to achieve a goal.

- Problem solving is central to mathematics and requires the use of prior knowledge and skills to deal with novelty, to overcome obstacles, to reach and validate solutions, and to pose problems.

- It is said that problem solving is at the heart of mathematics. We cannot imagine mathematics without problem solving (Ronda, 2010).

# Components of a Problem

A problem has three main components;

- an initial state (the current situation)
- a goal (the desired outcome) and
- a path for reaching the goal (including operations or activities that move you toward the goal)

## Features of a Problem

A problem has the following features:

- It must be perplexing and not simply a question
- It must be accepted by the individual who is faced with the task or situation.

- At the time it is presented to the student (or whoever is faced with the situation) there is some blockage or a challenge so that the solution is not known immediately.

A good problem must pass the following task potential test:

- What is problematic about the task?
- Is the mathematics interesting?

- What mathematics goals does the task address (and are they aligned with what you are seeking)?
- What strategies might students use?
- What key concepts and/or misconceptions might this task elicit?
- A good task should elicit more than one problem solving strategies like visualizing, looking for pattern, predicting and checking for reasonableness, formulating conjectures and justifying claims, etc
- A good task should have a;
- high cognitive demand,

- multiple entry and exit points, and
- relevant contexts
- Note from the foregoing features that a good problem is one that interests the problem solver and one that the problem solver makes an attempt to work.
- A worthwhile mathematics problem usually appeals to students and causes them to want to solve it.
- Good problems will integrate multiple topics and will involve significant mathematics (NCTM, 2000). They are mostly open-ended.

# DISCUSSION

**Example 1**

***Measure Problem:***

# DISCUSSION

Assume that you have a 3-litre measure and a 5-litre measure. How could you measure out 4-litres?

**Example 2**

***Eight Discs Problem:***

# DISCUSSION

There are 8 discs. Seven of them weigh the same, but one is just a bit heavier. Using a balance scale, how can you find the heavier disc in just **two** weighing?

**Example 3**

***Thickness of a single sheet of paper:***

# DISCUSSION

How do you find the exact thickness of a single sheet of paper? [Try finding the thickness of a pile of sheets and then divide by the number of sheets].

**Example 4**

**Three Guards in an orchard:**

Three guards were protecting an orchard. A thief met the guards, one after the other. To each guard

# DISCUSSION

he gave half the apples he had at the time and two extra. Eventually he escaped with just one apple. How many apples did the thief originally take?

**Example 5**

***The School-Bus Problem:***

When the sixth grade classes went on a trip, one of the two buses broke down and the children had to crowd on the seats of the second bus. The

# DISCUSSION

children were distributed in the bus in equal crowdedness. Three children were sitting on every 2 seats. The back seat of that bus had 4 seats. How many children were sitting on it?

# Definition of Problem Solving in Mathematics

- According to George Polya, problem solving is the process of finding the unknown means to a distinctively conceived end.
- If the end by its simple presence does not instantaneously suggest the means, we have to search for the means reflecting consciously how to attain the end; we have to solve a problem.
- To solve a problem is to find a way, where no way is known off-hand, to find a way out of a difficulty, to find a way around an obstacle, to

attain a desired end that is not immediately attainable, by appropriate means.

- Problem solving is defined as formulating new answers, going beyond the simple application of previously learned rules to achieve a goal.
- It is what happens when no solution is obvious.
- Problem solving should involve the use of open-ended approach where an “incomplete” problem is first presented to the problem solver and they are then encouraged to use many different approaches which may lead to a variety of correct responses (Becker & Shimada, 1997).

- The many correct answers are then used by the teacher to help students to construct new mathematical knowledge considering their previous knowledge, skills and ways of thinking.
- Problem solving has long been recognized as one of the most important mathematical processes (NCTM, 1980).
- It was recommended to be the focus of school mathematics at all the levels.
- According to NCTM Curriculum and Evaluation Standards, problem solving should be a primary goal of all mathematics instructions and

students should be encouraged to use problem solving approaches to investigate and understand mathematical content (NCTM, 1989).

- Solving problem is not only a goal of learning mathematics but also a major means of doing so.
- Problem solving is an integral part of all mathematics learning and not an isolated part of the mathematics programme (NCTM, 2000).

# Levels of Understanding in Mathematics

Our depth of understanding in mathematics is a continuum or hierarchy covering 7 levels: 1) Illiterates in mathematics,

2) Doers of mathematics,
3) Computers of mathematics,
4) Consumers of mathematics,
5) Problem solvers in mathematics,
6) Problem posers in mathematics, and 7) Problem creators in mathematics.

# Levels of Understanding in Mathematics

**CONSUMER LEVEL**

- Students at the consumer level of the continuum or hierarchy of understanding are those who can apply mathematical concepts to solve everyday problems.
- Such students are more functional in society; they are able to use mathematics every day, at work, home, in stores, businesses and in a multitude of other situations.
- Problem solving should aim at producing students who use mathematics to be intelligent consumers in society and to solve

# Levels of Understanding in Mathematics

everyday problems that require more than one step; that requires them to reason and plan a series of steps in order to solve routine and non-routine tasks.

**PROBLEM SOLVERS**

- They can apply their knowledge of mathematics in new situations when and where the answer is not obvious and they have no preset rule to fall back on.
- They try to use a method they have never used before or resort to an approach they have used to solve a different type of problem.

# Levels of Understanding in Mathematics

**PROBLEM POSERS**

- Problem posing is the ability to create, define, or pose problems and to see the important aspects of a situation and ask questions about it.
- This is where students develop mathematical power.
- They control their ability to learn mathematics by always questioning their results and searching for new understanding.

# Levels of Understanding in Mathematics

**PROBLEM CREATORS**

- They have the ability to create new questions upon which to work and then discover or invent mathematics to answer the questions.
- Even young children should be exposed to discovering or creating mathematics that is new to them.
- Students will understand and remember the mathematics they have constructed for themselves much better than any of the mathematics teachers try to teach them and search for new understanding.

# Levels of Understanding in Mathematics

**NB:** ***Problem posing*** *and* ***creating*** *can be classified under mathematical investigation.*

# Levels of Understanding in Mathematics

- Real-life problems are not always closed, nor do they necessarily have only one solution.
- To determine the best approach for solving a problem when several approaches are possible is a skill frequently required in everyday life.
- The solutions to problems, which are worth solving, rarely involve one item of mathematical understanding or just one skill.
- Rather than remembering a single correct method, problem solving requires students to search for clues and make connections to the various

pieces of mathematics and other knowledge and skills which they have learned. Such problems encourage thinking rather than mere recall.

# Definition of Mathematical Investigation

- In mathematics, proper investigation is often the first step in successful problem solving.
- Investigation helps to bring to the fore an essential feature of the subject itself.
- In ordinary language, to investigate is to ‘carefully examine a crime, a problem, a statement, etc., in order to discover the truth’.

# Mathematical Investigation

- Investigation involves trying things out with the aim of finding a solution to a problem
- Mathematicians do not usually spend all their time solving routine problems to obtain accurate and perfect answers.
- Many times mathematicians encounter situations they have never met before and for which there is no readily available means of solution.
- There is no readily known technique or tool to apply and they need to spend a great deal of time making trial and error and at times getting solutions that do not fit. This is termed investigation.

- Investigation tasks appear to be unfamiliar and so are the pathways to the solution

# Mathematical Investigation

- To Bailey (2007) mathematical investigation is “an open-ended problem or statement that lends itself to the possibility of multiple mathematical pathways being explored, leading to a variety of mathematical ideas and/or solutions” (p. 103).

# Mathematical Investigation

- In mathematical investigation, students do not necessarily know what is asked and do not know a direct way of solving it.
- When they finally arrive at a solution, they will have learnt a great number of useful ways of approaching problems.
- These useful ways are called ***strategies*** and with time they get answers much quicker than the average person.
- Investigation is a mathematical process often involving abstracting, generalizing, classifying and leading to problem solving, problem posing and problem creating.

- Mathematical investigations are activities that involve exploration of open-ended mathematical situations.
- The student is free to choose what aspects of the situation he or she would like to do and how to do it.
- Students pose their own problem to solve and extend it to directions they want to pursue.

# Mathematical Investigation

- Investigations often have some constraints in the form of rules to be followed and in this sense, they are similar to games.
- Small investigations can be used to introduce each new topic or use further investigation to develop understanding of concepts.
- Teachers should not walk students through every step of a problem for fear that they will be frustrated. Teachers should allow students to struggle and figure out how to learn on their own.

# Mathematical Investigation

- A good investigative task must be worthwhile. Worthwhile tasks should be problematic and pose questions for students.
- They usually:

  -have no prescribed or memorized rules or methods to solve

  -do not have a perception that there is one “correct” solution

  -offer boundaries or constraints

  -have the potential to provide new mathematical insights or knowledge

# Mathematical Investigation

- At the introductory stage of this unit we mentioned the seven levels of depth of understanding in mathematics which, places mathematics illiterates, doers, computers and consumers of mathematics at the lower levels of the hierarchy.
- Problem posers and creators are at the last two highest levels of the continuum amd can be classified under mathematical investigation.
- The key points about investigation are that students are encouraged to make their own decisions about:

# Mathematical Investigation

a)Where to start

b)How to deal with the challenge

c)What mathematics they need to use

d)How they can communicate this mathematics

e)How to describe what they have discovered.

# Mathematical Investigation

- Investigations are not necessarily extensive pieces of work.
- It could be a teacher's response to a question posed in a class.
- For example, in teaching about quadratic expressions as "algebraic expressions of the form " in high school and a student asks whether or not and are also quadratic expressions you can either ask the whole class to pursue this. In this case we say you are using investigation as a medium for learning.
- This allows students to think and discover things for themselves and this is more pleasing to students than to be told answers.

- The understanding students get in this way helps them to develop into “good learners” and this means that the pace of learning and capacity for learning will increase with time.

# Mathematical Investigation: Challenges

- One challenge of mathematical investigations is that there seems to be no specific problem to pursue or a clear path to follow.
- In this sense, mathematical investigation is a process-oriented mathematical activity.

- Students have the opportunity to choose what aspects of the situation they would like to do and what strategies to use to search for patterns, pose a problem, and state, and prove conjectures.
- This makes investigation activities to take much longer time to achieve than other mathematical activities.

# Mathematical Investigation: Example

Figure 1

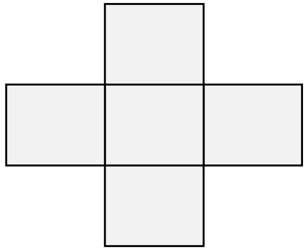


Figure 2

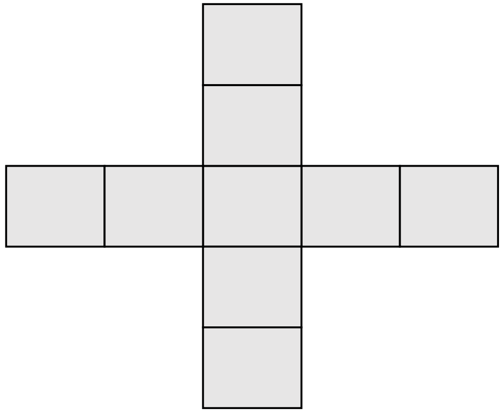


Figure 3

. . .

**Investigate.**

The student can investigate the number of squares in the nth figure, the perimeter of the figure in the nth figure, etc.

# Mathematical Investigation

- Although students may do the same mathematical investigation it is not expected that all of them will consider the same problem from a particular starting point.
- The ‘open-endedness’ of many investigations also means that students may not completely cover the entire situation.

- However, at least for a student's own satisfaction, the achievement of some specific results for an investigation is desirable (Ronda, 2010).
- Investigations help them build thinking skills, such as reasoning and problem solving, which can be applied to all areas of study.

## Problem Solving and Investigation compared

# **Discussion**

# Compare Problem Solving and Investigation in Mathematics

Problem Solving and Investigation in Mathematics

The following hugging tasks distinguish between problem solving and investigative tasks.

(i)At a workshop, each of the 100 participants hugs each of the other participants once. Find the total number of hugs. (ii)At a workshop, each of the 100 participants hugs each of the other participants once. Investigate.

# Comparison of Problem Solving and Investigation in Mathematics

- Task (i) is a problem solving task because it is closed, there is one question to solve, and it requires problem solving strategies to solve, such as, drawing a diagram for smaller number of participants to see if there is any pattern.

- In trying to find the total number of hugs the students are engaged in problem solving.

# Comparison of Problem Solving and Investigation in Mathematics

- Task (ii) is a rephrase of task (i) which now allows students to pose different problems to solve.

- For instance, what will it be for n participants? Or what if they hug each other m times? Here the students are doing mathematical investigation.
- In doing this investigation, students may at one time find the total number of hugs for 100 participants using the same heuristics.
- This suggests that in some instances, mathematical investigations and problem solving are synonymous; in other instances investigation can occur in closed problem solving tasks and not just in open investigative tasks.
- Investigation does not depend on whether a task has a closed or open goal.

# Mathematical Investigation: Example 1

Observe the following operation on numbers:

1. How do you think it works?

2. Every time this is done the answer is always 1089.

3. Investigate the trick.

| | | | | |
|---|---|---|---|---|
| | 8 | 4 5 | | 4 |
| □ | 3 | 5 | | 8 |
| | 6 | 9 | | 6 |
| □ | 0 | 9 | | 3 |
| 1 | | 8 | | 9 |

# Mathematical Investigation: Example 1

- Though this task has a closed goal since students are to investigate the trick and not to select any other goal to investigate, it can still be considered as an investigation.
- Several questions are posed by students to determine where to start from and for extension.
- Does this work for other numbers?
- Which kind of numbers does it work for?
- Should it always be three-digit numbers?

- Try many other pairs of numbers.
- Discuss your results.

# Mathematical Investigation: Example 2

**Frogs and Toads (Re-visited)**

| | | | | | | |
|---|---|---|---|---|---|---|
| F | F | F | | T | T | T |

- There are three Frogs and three Toads arranged as shown. There is an empty space separating them.
- Can you help the frogs and the toads change places using the following rules:
  a) A frog or a toad can slide into an empty space.

# Mathematical Investigation: Example

b) A frog or toad can hop over one toad or one frog into an empty space.

2

- Think of a way of recording your moves.
- What is the smallest number of moves you can make?
- Try changing the number of frogs and toads and see what happens.
- See if there is a rule to use.

# Mathematical Investigation: Example 3

**Calculation conundrum**

- Ask a partner to do the following:
    1. Write down any 2 single-digit numbers, one under the other. E.g., 3 and 4.
    2. Add them and write the sum underneath as the third number.
    3. hen add the second and the third numbers and write the sum underneath as the 4th number.

# Mathematical Investigation: Example

4. Then add the third and the fourth and write underneath as the 5th number.
5. Continue doing this until there are 10 numbers in the column.

# Mathematical Investigation: Example 3

**Calculation conundrum**

6. While your partner does this keep your back turned, do not watch.
7. After the 10 numbers are completed turn around and immediately write down the sum (605, for this example) of the 10 numbers by just looking at a few of the ten numbers. Ask your partner to add all ten numbers and confirm your answer.

# Mathematical Investigation: Example 4

- Can you explain how this is done so fast? Does this work always?

**The Bill** $3$

$.1\ 9$

This bill has the correct total, but the digits of the three items have been written down in the wrong order. For example, 3.19 could be 1.39 or 3.91.

Rearrange the items so their sum will be 8.73.

# Mathematical Investigation: Example 5

$6 . 4 7 8 . 2 5$ $\qquad\qquad\qquad$ $8 \quad . \quad 7 \quad 3$

**Polite numbers**

- The number 15 can be expressed as a sum of two consecutive natural numbers, 7+8 or as the sum of three consecutive natural numbers, 4+5+6.
- Natural numbers that can be expressed as the sum of two or more consecutive natural numbers are said to be polite numbers.

# Mathematical Investigation: Example 6

- Can all counting numbers be expressed as the sum of two or more consecutive counting numbers? If not, which ones can?

***Experiment, look for patterns, and come up with some conjectures.***

***Write up what you find.***

# Mathematical Investigation: Example 7

**Arrangement of Numbers Problem**

The numbers 1 to 9 have been arranged in a square so that the second row, 384, is twice the top row, 192. The third row, 576, is three times the first row, 192.

| | | |
|---|---|---|
| 1 | 9 | 2 |
| 3 | 8 | 4 |
| 5 | 7 | 6 |

# Mathematical Investigation: Example 8

***Arrange the numbers 1 to 9 in another way without changing the relationship between the numbers in the three rows.***

**Wolf, Goat and Cabbage**

You are travelling through a difficult country, taking with you a wolf, a goat, and a cabbage. All during the trip the wolf wants to eat the goat, and the goat wants to eat the cabbage, and you have to be careful to prevent either calamity. You come to a river and find a boat which can take you across, but it's so small that you can take only one passenger

# Mathematical Investigation: Example 9

at a time – either the wolf, or the goat, or the cabbage. You must never leave the wolf alone with the goat, nor the goat alone with the cabbage.

1. *How can you get them all across the river?*
2. *How many trips across the river will there be before the crossover is complete?*

# Mathematical Investigation: Example 10

**Four Operations Problem**

- Put all the numbers 1 to 9 in the boxes so that all four operations are correct. $\square - \square = \square$
- Fill in the boxes with a different set of $\times$ numbers so that the four equations are $\square \div \square =$ still correct

$$\square$$

$$||$$

$$\square + \square = \square$$

# Mathematical Investigation: Example 11

**Dwarf and river crossing**

- In a land of dwarfs, a mysterious dwarf decides to visit three of her friends. She carries seven (7) mangoes in her sack. To reach her first friend she has to cross a magic river. After she crosses the river, the number of mangoes in her sack doubles. She then gives her first friend a number of mangoes. She continues her journey and then crosses the second magic river. The number of mangoes doubles again. She gives her second friend the same number of mangoes. She crosses the third magic river. Again the number of mangoes doubles. She visits her third

# Mathematical Investigation: Example 12

friend and gives the same number again. This time there are no mangoes left.

***What number did she give to each friend?***

# Mathematical Investigation: Example 10

$y$ $x$ $x$ $x$

Investigate with different

start numbers. $y$ and $y$ $y$ $y$ ***What numbers work out*** □ $y$ □ $z$

***Z Z neatly?*** ______________________

$b$ $y$

$a$ $b$ $c$ $d$

# Values of Teaching through Problem solving

**1. Problem solving places the focus of the students' attention on ideas and sense making.**

- When students are solving problems they reflect on the ideas that are inherent in the problems.
- These ideas are more likely to be integrated with existing ones, and this improves understanding.

# Session 5: Values of Teaching through Problem solving

2. **Problem solving develops "mathematical power".**
   - Students are engaged in the five mathematical processes of "doing mathematics" – problem solving, reasoning, communication, connections, and representation.
3. **Problem solving develops the belief in students that they are capable of doing mathematics and that mathematics makes sense.**

- Every problematic task from the teacher indicates his/her belief in the students’ ability to do it.
- Every problem solved by students builds their confidence and selfworth.

## Values of Teaching through Problem solving

4. **Problem solving provides ongoing assessment data that can be used to make instructional decisions, help the students succeed, and inform parents.**

- Students discuss ideas, defend their solutions and evaluate others', and write reports or explanations.
- These provide valuable information for planning the next lesson, helping individuals, evaluating their progress, and communicating with parents.

5. **It is a lot of fun.**
   - The excitement of students developing understanding through their own reasoning is worth all the effort. This is fun for the students.

# Values of Teaching through Mathematical Investigations

Teaching mathematics through investigations has the following benefits:

1. It develops students' mathematical thinking processes and mental habits.
2. It deepens students' understanding of the content of mathematics, and challenges them to "produce" their own mathematics within their universe of knowledge.

3. Integrating Mathematical Investigations in the mathematics classes is a way of encouraging schools to focus on the learners' reasoning, communicating and problem solving skills and processes.

## Values of Teaching through Mathematical Investigations

4. Students take responsibility for their own learning and become autonomous learners. They acquire integrated understanding of concepts, pose important questions and then find answers to the questions.

5. Additionally, through investigations, students gain insight into cultural practices of mathematicians, and mathematics as a career.

# UNIT 2

# PROBLEM SOLVING AND INVESTIGATION STRATEGIES

# Unit Outline

Heuristics

George Polya’s Model

John Mason’s Model

The ‘IDEAL’ and Wood’s Models

The Six- Step Model and Harrrison’s models Characteristics of Good Problem solvers

# Heuristics

- A ***heuristic*** is a general suggestion or strategy independent of any particular topic or subject matter that helps problem solvers approach and understand a problem and efficiently marshal the resources to solve it.
- ***Heuristics*** are general approaches, strategies or abilities which are helpful in solving problems and investigations.

- ***Strategies*** are tools that might be useful in discovering or constructing means to achieving a goal.

# Strategies used in problem solving and investigations

The strategies used in problem solving and investigations are determined by two factors:

1. The skill and sophistication level of the problem solver or investigator.
2. The range of mathematical tools (e.g. content learned, familiarity with similar problems, etc.) that the problem solver or investigator has mastered.

- making a model,

# Strategies used in problem solving and investigations

- drawing a diagram or a graph,
- making a table,
- selecting a notation,
- looking for a pattern,
- guessing and checking,
- restating the problem,
- acting it out,

# Strategies used in problem solving and investigations

- looking at all possibilities (systematically),
- working backwards,
- solving simpler problem with less variable,
- checking for hidden assumption,
- relaxing a condition and then try to re-impose it,
- decomposing the problem and then working on it case by case,
- exploiting any previous problem with similar form, givens or conclusions,

# Strategies used in problem solving and investigations

- using trial and error

# Heuristics used in problem solving

(a) Selecting appropriate notation
(b) Drawing a diagram or graph
(c) Identifying wanted, given and needed information
(d) Restating the problem
(e) Writing an open sentence
(f) Constructing a table

# George Polya's Model

George Polya, a noted scholar in the area of problem solving theory, indicated that solving a problem is finding the unknown means to a distinctly conceived end (Polya, 1957).

George Polya developed a four-step problem solving process. These are:

(i) Understanding the Problem

- what am I solving for?
- What do I know about the problem in terms of what is given?
- Are there any possible hints or particular terms that need to be defined?

# George Polya’s Model

- What additional information do I need?

**(ii) Devising a Plan**

- Is there a formula to use?
- Do I need to draw a table or diagram?
- Is there an identified pattern?
- Have I solved a similar problem before?
- What relevant previous knowledge do I need in order to solve this problem?

# George Polya’s Model

**(iii) Carrying out the Plan**

- Once you know how to approach the problem, carry out your plan.
- You may run into unforeseen roadblocks, but be persistent.
- There is also the possibility that the plan may not work out and may call for modification(s).
- Do the modification(s) and restart the process.

**(iv) Looking Back**

- Check your answer to see that it is reasonable. (Imagine having an answer of one-half human being).

# George Polya’s Model

- See if it satisfies the conditions of the problem.
- It may call for substituting your answer into the given problem to see if it works especially when dealing with equations to be sure that the values obtained satisfy the equations produced.
- Ensure that you answered all the questions the problem asks.
- Determine that you did not misunderstand the question or the conditions given.

# Using Polya’s Problem solving Process

**Problem:**

When the famous German mathematician Carl Fredrich Gauss was a child, his teacher required the students to find sum of the first 100 natural numbers.

The teacher expected this problem to keep the class occupied for some time. Gauss gave the answer almost immediately. The explanation given below shows how he went about solving the problem:

## Using Polya's Problem solving Process

**Understanding the Problem.**

- The natural numbers are 1, 2, 3, 4, …

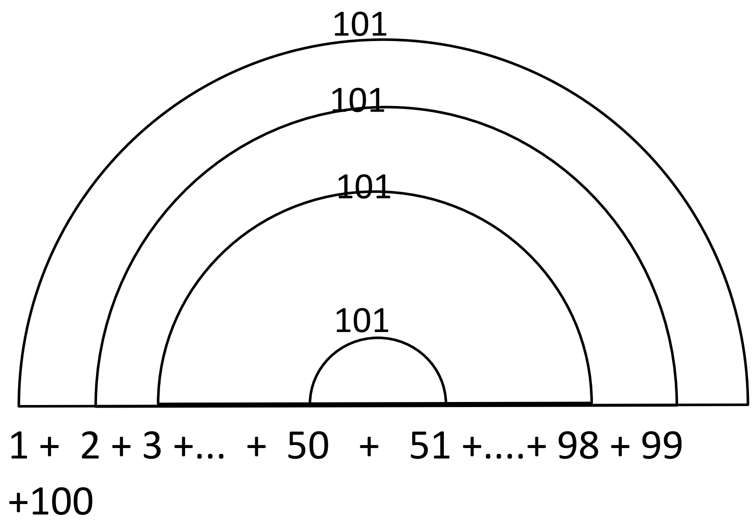
- Thus the problem is to find the sum $1 + 2 + 3 + 4 + \ldots + 100$.

# Using Polya's Problem solving Process

**Devising a Plan**

- One possible strategy is that of looking for a pattern.
- The obvious way of adding the numbers in the order they appear does not reveal a recognizable pattern.

- However, by considering 1 + 100, 2 + 99, 3 + 98, …, 50 + 51, it is evident that there are 50 pairs of numbers, each with a sum of 101, as shown below.



# Using Polya's Problem solving Process

**Carrying out the Plan**

- There are 50 pairs, each with the sum 101. Thus the total sum is 50(101) = 5050.

**Looking Back**

- The method is mathematically correct because addition can be performed in any order, and multiplication is repeated addition.

- A more general problem is to find the sum of the first n natural numbers:

$$\frac{(\ +1)}{2}$$

- 1 + 2 + 3 + 4 + 5 + … + n is given by

# John Mason’s Model

John Mason’s model provides a process for tackling a problem. It consists of three phases called ***Entry, Attack and Review***.

**The Entry Phase**

- What do I know?
- What do I want?
- What can I introduce?

# John Mason's Model

**The Attack Phase**

- Thinking enters the Attack phase when you feel that the problem has become your own.
- The mathematical activities that might take place in the Attack phase are complex and varied.
- The states which are particularly associated with the Attack phase are 'STUCK!' and 'AHA!' and the fundamental mathematical processes called upon are conjecturing and justifying convincingly.

# John Mason’s Model

- During the Attack phase several different approaches may be formulated and tried out.

**The Attack Phase**

- When a new plan is being implemented, work may progress at a great rate.
- On the other hand, when all ideas have been tried, long periods of waiting for new insight or for a new approach may characterize the phase.

# John Mason’s Model

**The Review Phase**

- When you reach a reasonably satisfactory resolution, when you are about to give up or at the end of your work, it is essential to review your work.
- As the name suggests it is a time for looking back at what has happened in order to improve and extend your thinking skills, and for trying to set your resolution in a more general context.

# John Mason’s Model

- It involves both looking back, to CHECK what you have done and to REFLECT on key events, and looking forward to EXTEND the processes and the results to a wider context.

# ‘IDEAL’ Model

The five stages of the ‘IDEAL’ model are:

- (i) Identify the problem and opportunities;
- (ii) Define goals and represent the problem;
- (iii) Explore possible strategies; • (iv) Anticipate outcomes and Act; and
- (v) Look back and Learn.
- ***Identify*** that a problem exists (a blockage) and treat it as an opportunity.

# ‘IDEAL’ Model

- This is the critical first step to begin the process.

- The ***define*** stage requires you to set goals and be sure of what the problem demands.
  - This will also require you to determine what the end point is and try to model the problem using the information given.
- The ***explore*** stage involves identifying possible strategies worth trying and decide on where to take off.

# ‘IDEAL’ Model

- Get started when you are certain on a particular strategy, working toward an anticipated endpoint.

- The ***anticipated*** outcome simply means determining what the task is.
  - What is it I am solving for?
    - How do you I know that the problem is solved without going on and on?
- Finally ***look back and learn***.

# ‘IDEAL’ Model

- Check your solution by testing the result, going over the process used.
  - Do extension where necessary to consolidate and build on it. Review the work done.

# Characteristics of Good Problem Solvers

- Good estimation and analytical skills
- Ability to perceive sameness and differences
- Reflective and creative thinking
- Ability to visualize relationships
- Strong understanding of concepts and terms
- Ability to disregard irrelevant data

# Characteristics of Good Problem Solvers

- Capability to switch methods easily
- Ability to generalize on the basis of few examples
- Ability to interpret quantitative data
- Strong self esteem
- Low test anxiety

# UNIT 3

# THEORIES AND PRINCIPLES OF PROBLEM SOLVING IN MATHEMATICS

## UNIT OUTLINE

- Teaching For Problem Solving,

- Teaching About Problem Solving,
- Teaching Through Problem Solving,
- Role of the Teacher in Teaching Problem Solving,
- Challenges in Teaching Problem Solving

# Principles Guiding Teaching Problem Solving

1. Model a useful problem solving method.
2. Teach within a specific context.
3. Help students understand the problem.

4. Take enough time.

5. Ask questions and make suggestions.

6. Link errors to misconceptions.

## Teaching For Problem Solving

- In teaching for problem solving, the teacher presents the mathematics, the students practise the skills and finally, students solve story problems that require using the skill.

- In the past, we assumed that walking the students through a procedure or showing a step-by-step method for solving a particular type of problem was the most helpful approach to learning.

## Teaching For Problem Solving

- It has been noted by many mathematicians and other scientists as well as lay people that unfortunately this way to the teaching of mathematics has not seen the light of day for majority of learners in understanding various concepts in mathematics.

- For a student to be able to figure out an approach to solve the problem at hand is really what it means to do mathematics.

## Teaching For Problem Solving: Reasons why students are not able to understand mathematics through this means

- This approach to mathematics teaching requires that every student possess the needed requisite knowledge to be able to understand what the teacher puts forward which mostly doesn't happen.

- It makes one believe that there is only one way to solving any particular problem which in a way misrepresent the field of mathematics and in a way disempower students who may naturally want to “outside the box” to try a new way of solving the problem.

## Teaching For Problem Solving: Reasons why students are not able to understand mathematics through this means

- The learner is placed in a passive position always depending on the teacher to present his/her ideas rather than seeing themselves (students) as self-determining scholars who have all it takes to solving the problem presented to them.

- It reduces the possibility a student will try a fresh problem without obvious instruction on how to solve it. The student end up by saying that “but that’s what doing mathematics is-figuring out an approach to solve the problem at hand.

# Teaching About Problem Solving

- Teaching about problem solving means proving a ‘guide’ or ‘how-tosolve’ to students about how to engage in problem solving tasks.
- Although such strategies or guides may be devoid of mathematical content it is expected that they are generally transferrable to lots of school mathematics tasks and that such thinking are also useful in solving everyday life problems.

# Teaching About Problem Solving

When we teach students various problem solving strategies

- We are engaging in teaching about problem solving.
- We are providing you with some strategies we believe can help you solve future problems such as;
  - ☐ there is always a need to understand the task at hand,
  - ☐ identifying and choosing an appropriate and efficient solution strategy among others.

- The expectation is that by teaching you these strategies you will learn to think like a mathematician (think mathematically) even when you encounter new tasks that you have no immediate solution strategy for.

## Common Problem solving strategies in school mathematics

- Visualize
- Looking for a pattern
- Predict and check for reasonableness

- Formulate conjectures and justify claims Van de Walle, et al (2011)

# Teaching Through Problem Solving

- Teaching for and about problem solving approaches indicate that transfer of learning (skills) is not straightforward.
- As such, if the goal of teaching mathematics is for students to learn to think mathematically through engaging in problem solving then problem solving tasks should be at the forefront of our teaching not at the backend.

# Teaching Through Problem Solving

- In this regard, teaching through problem solving can be seen as the reverse of teaching for problem solving.
- Teaching through problem solving according to Van de Walle, et al. (2011) “might be described as upside down from teaching for problem solving – with the problem or task presented at the beginning of a lesson and related knowledge or skills emerging from exploring the problem” (p.37).

# Teaching Through Problem Solving

- In “teaching through problem solving,” the goal is for students to learn precisely the mathematical idea that the curriculum calls for them to learn.
- A “teaching through problem solving” lesson would begin with the teacher setting up the context and introducing the problem.

# Teaching Through Problem Solving

- Students then work on the problem for about 10 minutes while the teacher monitors their progress and notes which students are using which approaches.
- Then the teacher begins a whole-class discussion
- In teaching through problem solving as prospective teachers, our style of teaching should be the exact opposite of teaching by telling.
- As a teacher you do not do the explaining.

# Teaching Through Problem Solving

- Develop the belief that students are capable of doing math and making sense of mathematics.
- One approach is to organize instruction using the Three-part lesson format ~ the “Before, During and After” model (refer to PCK in mathematics).

# Teaching Through Problem Solving

- Also, teaching through problem solving basically means students learn mathematics through inquiry.
- Students are allowed to explore real context, problems, situations, and models, and from those explorations they learn mathematics.
- Modelling a useful problem solving method
- Teaching within a specific context
- Helping students understand the problem

# ROLE OF THE TEACHER IN TEACHING ROBLEM SOLVING

- Taking enough time
- Asking questions and making suggestions
- Linking errors to misconceptions
- Communicating
- Encouraging independence
- Being sensitive
- Encouraging thoroughness and patience
- Brainstorming: Invite students to be fluent thinkers by asking them to respond to questions that have many right answers

# ROLE OF THE TEACHER IN TEACHING ROBLEM SOLVING

- Reflecting: Help students to be flexible thinkers by asking them to comment on specific objects or situations in your room
- Providing plenty of time every day for students to choose activities based on their interests and developmental levels
- Following students' leads.
- Reinforcing students' solutions
- Extending creative thinking and problem solving
- Stepping back and watching students' independent problem solving
- Focusing on the process students are engaged in

# ROLE OF THE TEACHER IN TEACHING ROBLEM SOLVING

- Acknowledging students' efforts, letting them know that what they are doing is important
- Creating accepting environments where students feel free to express their ideas without fear of being wrong
- Giving students opportunities for open ended play activities in long periods of time
- Providing a variety of problem solving experiences

# Challenges in Teaching Problem Solving

- Teacher discomfort
- There is also that fear among teachers that students will come up with ideas that they (teachers) won't understand
- Student insecurity: This may occur because the students may not have met open-ended problems before.
- Curriculum constraints
- Low ability students

- Preparation time: Difficulty in finding the right problem to use to introduce a given lesson.

# UNIT 4

# Problem Solving and Investigation Tasks

# UNIT OUTLINE

- Warm Up Activities,
- Number and Algebra related problems,
- Geometry-Related Problems,
- Logic – Related Tasks

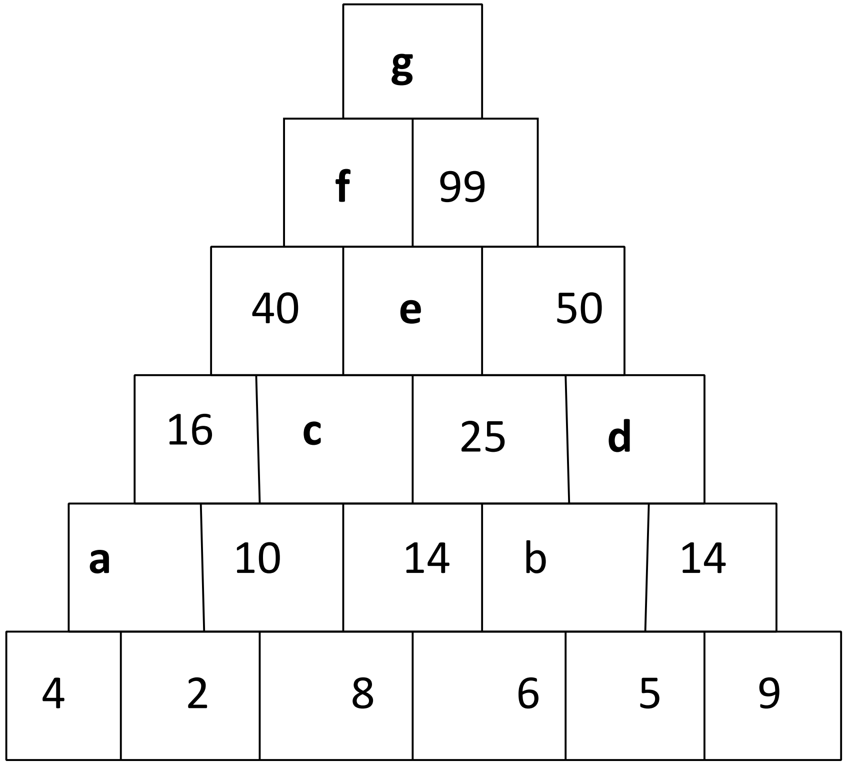
# Warm Up Activities

- Warm up activities are generally used to set the stage for learning.

- The pupils’ readiness to learn depends on a teacher’s ability to capture the minds of the pupils, arouse their interest and sustain their interest throughout the learning process.
- A good warm up activity or teaser will do this trick of arousing and sustaining the interest of the pupils throughout the learning process.

# Warm Up Activity 1

*Study the pattern displayed. Find the values of the letters.*




**Solution:**

$=6; =11; =24; =25; =49;$
$=88; =187.$

# Warm Up Activity 2

**Extension**: *Can you think of another way to determine the values of the letters?*

Two numbers add up to 6.

a) What are they?

b) How many different answers are there?

# Warm Up Activity 3

**Solution:** $(1 \quad 5; 2 \quad 4)$

$;(3 \quad 3).$

a)

b) Three different answers.

The number 24 has the property that it is one less than a square number, and its double is also one less than a square number. That is,

$$24 \square 1 \square 25 \square 5^2$$

# Warm Up Activity 4

$(2\square 24)\square 1\square 49\square 7^2$

Find the next number which has this same property as 24.

**Hint:** *Make a table of square numbers and investigate.*

[Check to see if 840 has this property]

**Safety First (Brian Bolt, 1982)**

Each letter stands for a different digit. S, for instance stands for

# Warm Up Activity 5

3. What do the other letters represent?

$$C \quad ROSS$$

$$\square\, ROADS$$

$$D \quad ANGER$$

$$SEND \square MORE$$

$$MONEY$$

# Warm Up Activity 6

**Pascal's Triangle**

You are familiar with this triangular array of numbers known as Pascal's triangle, named after the French mathematician and philosopher Blaise Pascal.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

a guess as to the sum of number in the □ □ □ □ □ □ □ □ □ □ □ 10th row.

# Warm Up Activit

1. Can you give the next two lines?

2. Find the numbers in each row and make $1 \quad 5 \quad 10\,10\,5 \quad 1$

y 5

**Pascal's Triangle** $1$

You are familiar with this triangular array of

a guess as to the sum of number in the           
10th row.

# Warm Up Activit

numbers known as Pascal’s triangle, named 1 1

after the French mathematician and 1 2 1 philosopher Blaise Pascal.

1 3 3 1

1 4 6 4 1

1. Can you give the next two lines?

a guess as to the sum of number in the □ □ □ □ □ □ □ □ □ □ □ 10th row.

# Warm Up Activit

1 5 10 10 5 1

2. Find the numbers in each row and make

y 5

**Pascal's Triangle**

1

You are familiar with this triangular array of

numbers known as Pascal's triangle, named

1 1

a guess as to the sum of number in the □ □ □ □ □ □ □ □ □ □ □

10th row.

# Warm Up Activit

after the French mathematician and 1 2 1 philosopher Blaise Pascal.

1 3 3 1

1 4 6 4 1

1. Can you give the next two lines?

1 5 10 10 5 1

2. Find the numbers in each row and make

   a guess as to the sum of number in the □ □ □ □ □ □ □ □ □ □ □
   10th row.

# Warm Up Activit

# y 5

This pattern occurs in many situations which you can investigate.

**Watch this out!!!**

1

1 1

1 2 1

110 □1 1 3 3 1

111 □1 11 4 6 4 1

112 □ 1 2 1   1   5   10 10 5   1

113 □   1 3   3 1□ □ □ □ □ □ □ □ □ □ □

Determine the stage at which the pattern ceases to look like Pascal's triangle and explain why.

# Number and Algebra related problems

- Working with numbers or number sense is crucial in the successful learning of mathematics.

- Number sense involves having facility with numbers by working with them flexibly and with understanding.

# Number and Algebra related problems: Primary School

**Activity 1:**

Fill in the empty spaces.

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 5 | | |
| | | | | |

**Solution**

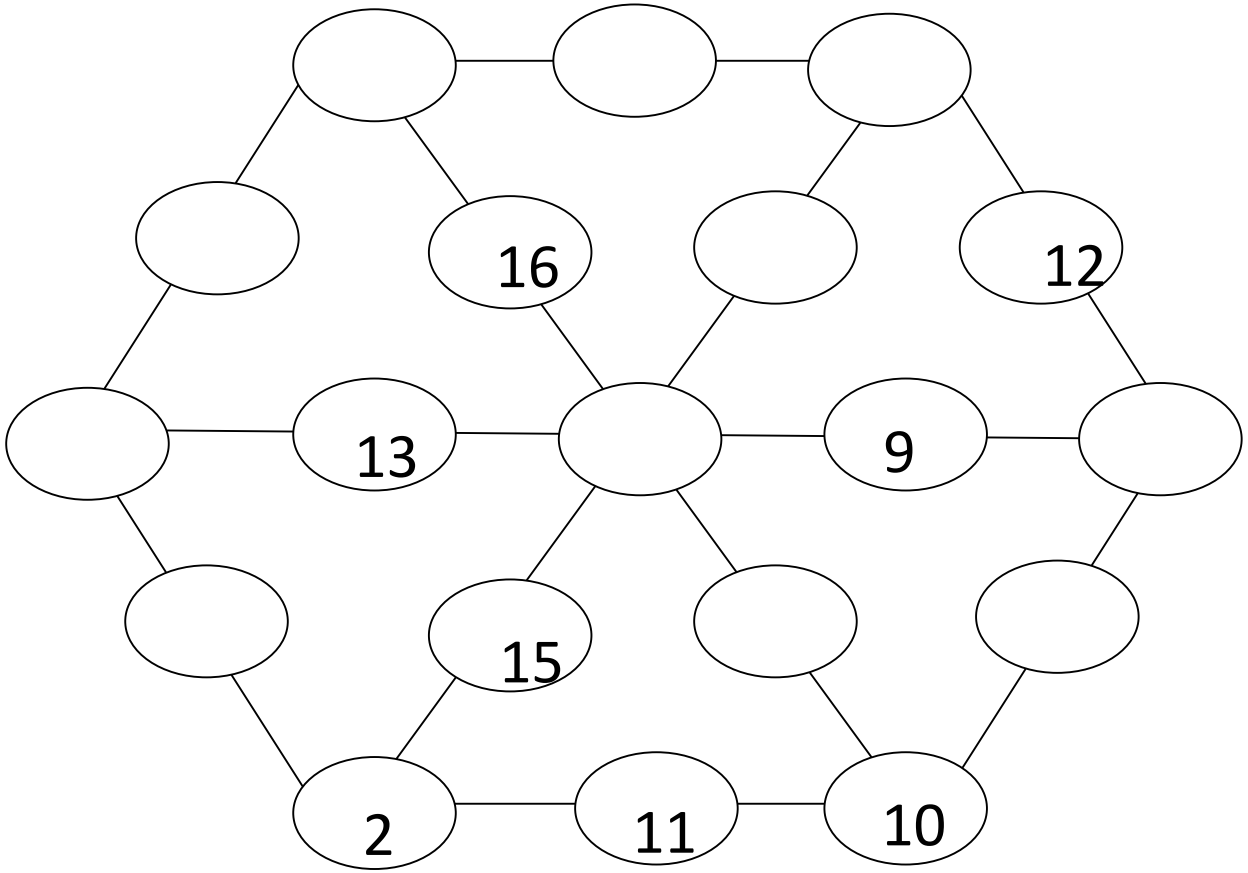
| | | | | |
|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 |
| 2 | 4 | 6 | 8 | 10 |

| | | | | |
|---|---|---|---|---|
| | | | 8 | 10 |
| 19 | 17 | | 13 | |
| 20 | | 16 | | 12 |
| 21 | | 25 | | 29 |

| | | | | |
|---|---|---|---|---|
| | | | | |
| 19 | 17 | 15 | 13 | 11 |
| 20 | 18 | 16 | 14 | 12 |
| 21 | 23 | 25 | 27 | 29 |

Number and Algebra related problems: Primary Sch

**Activity 2:**



# Number and Algebra related problems: Primary School

**Activity 2:**

The diagram shows a hexagon containing six triangles all sharing a vertex at the centre of the hexagon. There are three circles along each side of each triangle. Numbers in the three circles along each side/edge of the triangle must add to the same number.

a)What is the number?

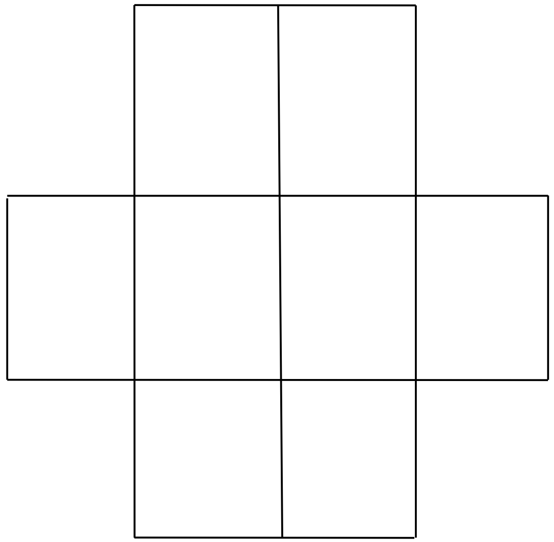
b)Find all the missing numbers.

c)Now arrange the numbers 1 to 19 in a similar wheel so that the total along each of the twelve lines is 22.

# Number and Algebra related problems: JHS Level

**Activity 1:**

Place each of the digits 1, 2, 3, 4, 5, 6, 7 and 8 in separate boxes so that boxes that share common corners do not contain successive digits.



**Solution:**

Investigate to find that one possible sequence: top row (4, 6) middle row (7,1, 8, 2), and bottom row (3, 5).

**Extension:** *Can you think of another set of numbers that will satisfy the condition?*

# Number and Algebra related problems: JHS Level

## Activity 2

Complete the following and Work out the following and explain explain the pattern. any patterns that emerge.

# Number and Algebra related problems: JHS Level

$143 \times 2 \times 7 = 143$

$\times 3 \times 7 =$

$143 \times 4 \times 7 = 143$
$\times 5 \times 7 = 143 \times 6$
$\times 7 =$

$(0 \times 9) + 1 =$

$(1 \times 9) + 2 =$

$(12 \times 9) + 3 =$

$(123 \times 9) + 4 =$

$(1234 \times 9) + 5 =$

**Activity 3:** Why is it so? (adopted from Sokpe & Osiakwan, n.d) Write down any three numbers less than ten, e.g. 2, 4 and 7.

# Number and Algebra related problems: JHS Level

- Make all the six possible 2-digit numbers using the three digits. E.g. 24, 27, 42, 47, 72, 74.
- Find their sum. (286)
- Calculate the sum of the original numbers: 2 + 4 + 7 = 13.
- Divide the first total by the second. (286 / 13)

**Answer? 22**

***Repeat the operations for other combinations of numbers.***

**Activity 4:** Zim and Zog. (Adapted from ‘I can solve it’)

# Number and Algebra related problems: JHS Level

Jane and Mo are playing computer game. The game needs them to share out gold coins between 2 giants, Zog and Zim. When there are 2 coins Jane says we can share out the coins in 3 ways.

- How did she do it?
- How many different ways can they do it for four coins?
- Find out the number of ways they can do it for 10 coins

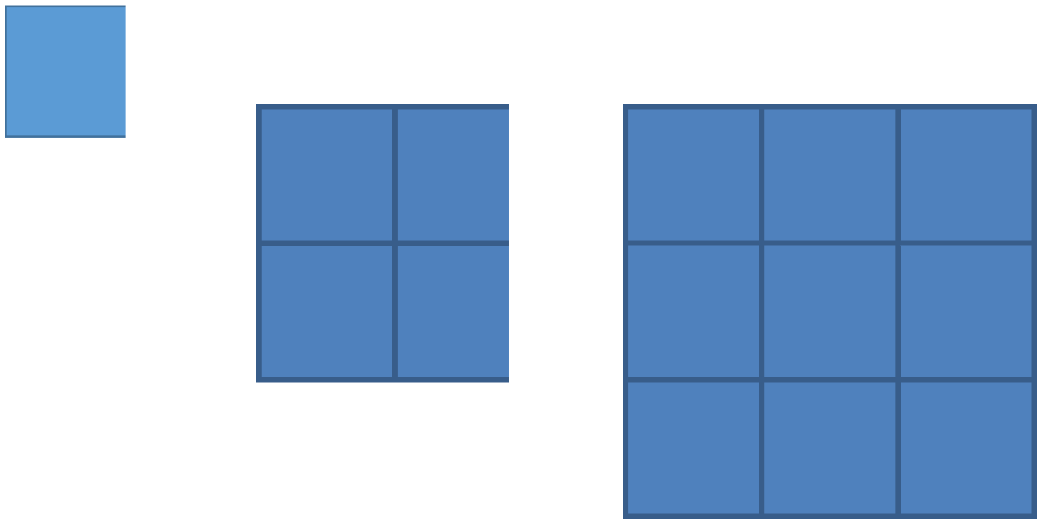
# Number and Algebra related problems: JHS Level

**Activity 5:** Making a Hundred

- Can you put ‘+’ and ‘ – ‘ signs in the following so as to make 100? 1 2 3 4 5 6 7 8 9?
- How many different ways can you do this?

**Activity 6:** Square Numbers
Look at the following patterns:

# Number and Algebra related problems: JHS Level



a)Predict the number of squares needed for the next two patterns?

b)How many squares will be needed for the 10th pattern?

# Number and Algebra related problems: JHS Level

| Position | Total number of Squares |
|---|---|
| 1st | |
| 2nd | |
| 3rd | |
| 4th | |
| 5th | |

# Number and Algebra related problems: JHS Level

**Activity 6:** Square Numbers

| | |
|---|---|
| 7th | |
| 10th | |

c)Copy and completethe table below.

*State a verbal rule to describe the relationship between the total number of squares and the position of a pattern*

**Activity 7:** Cuts, Folds and Pieces State:

# Number and Algebra related problems: JHS Level

a) A verbal algebraic relationship between the number of pieces and the number of cuts when an unfolded rope or twine is given a number of cuts.

b) A verbal algebraic relationship between the number of pieces, the number of folds and the number of cuts when a folded rope or twine is given a number of cuts.

**Activity 7:** Spring Flowers

# Number and Algebra related problems: JHS Level

On her breakfast tray, Aunt Lily had a little vase of flowers - a mixture of primroses and celandines. She counted up the petals and found there were 39. “Oh, how lovely!” she said, “exactly my age; and the total number of flowers is exactly your age, Rose!” How old is Rose?

(***NB: Primroses have five petals on each flower and Celandines have eight petals on each flower***).

# Number and Algebra related problems: JHS Level

**Activity 7:** Spring Flowers

**Solution:**

- The way to figure out is to start at the end with 39 petals, and remember that the primroses must provide either 5 petals or 10 or 15 or 2 or 25 or 30 or 35- a multiple of 5.
- Now suppose there was only one celandine (8 petals); that would leave 31 petals (39-8=31).
- The rest can’t be primroses, because 31 is not an exact multiple of 5. Suppose there were two celandines =24 petals and 39-24=15 petals. Bingo!
- The answer must be three celandines and three primroses. So Rose is (3+3), or

# Number and Algebra related problems: JHS Level

6.

# Number and Algebra related problems: SHS Level

**Activity 1:** Teenager’s age

A teenager’s age increased by 2 gives a perfect square. Her age decreased by 10 gives the square root of that perfect square. She is 5 years older than the brother. How old is her brother?

**Answer**: The teenager is 14 years old and her brother is 9 years old.

**Activity 2:**

# Number and Algebra related problems: SHS Level

A man put a pair of rabbits in a cage. During the first month the rabbits produced no offspring, but each month thereafter produced one new pair of rabbits. If each new pair thus produced reproduces in the same manner, how many pairs of rabbits will there be at the end of one year?

# Number and Algebra related problems: SHS Level

**Solution**

At the start of the first month there is only one pair of rabbits. No new pairs are produced during the first month, so there is pair present at the end of the first month. This pattern continues throughout the table.

# Number and Algebra related problems: SHS Level

| Month | Number of Pairs to Begin with | Number of New Pairs Produced | Number of Pairs at End of Month |
|---|---|---|---|
| 1st | 1 | 0 | 1 |
| 2nd | 1 | 1 | 2 |
| 3rd | 2 | 1 | 3 |
| 4th | 3 | 2 | 5 |
| 5th | 5 | 3 | 8 |

# Number and Algebra related problems: SHS Level

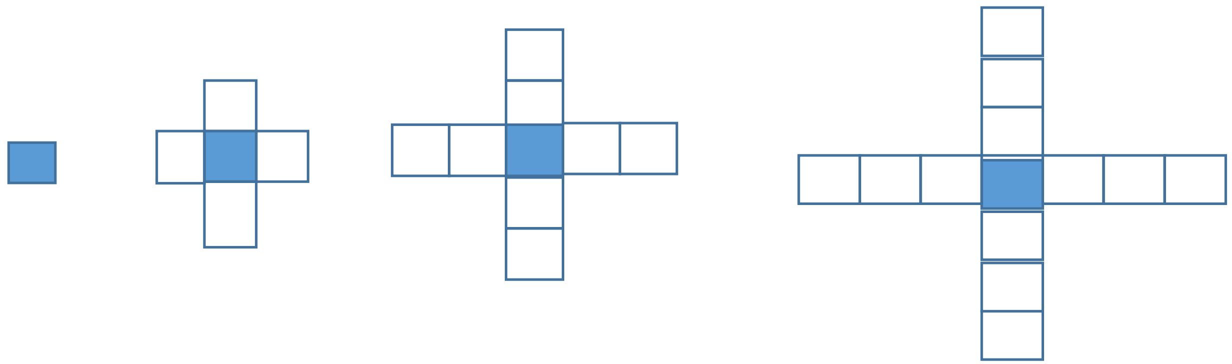
| | | | |
|---|---|---|---|
| $6^{th}$ | 8 | 5 | 13 |
| ……… | | | |
| ………. | | | |

See if you can continue the pattern to obtain the number of pairs at the end of year one.

**Activity 3**: Linear function

Look at the following patterns:

# Number and Algebra related problems: SHS Level



- Can you predict the number of blocks needed for the 10th pattern? Show your work.
- Now that you have done that can you determine the formula for the nth pattern?

# Number and Algebra related problems: SHS Level

**Solution**

- I trust that you obtained the formula for the nth pattern to be $y = 4x + 1$?

**Extension:**

- Since the formula for the nth pattern is a linear function, using either a graph or a table show clearly the coefficient of and 1.

# Number and Algebra related problems: SHS Level

- Also, explain what these values mean in relation to the growth pattern. Hint.

*Think of Binomial Theorem.*

**Activity 4a:** On the first day of EBS 417 class comprising 15 students, the course tutor asked every student to shake hands and introduce himself or herself to each other. How many handshakes would be exchanged at the end of the introduction? **Answer:** 105

# Number and Algebra related problems: SHS Level

**Activity 4b:**Can you determine the total number of handshakes that are possible if Aisha's class had 45 students? What about if there were students?

**Activity 4b Solution**

For 45 students, the number of handshakes will be 990.

For the general case, number of handshakes (h) will be given by; $h = \frac{n(n-1)}{n}$

# Number and Algebra related problems: SHS Level

**Activity 5:** The set of whole numbers is partitioned into subsets with the first number in the first subset, the next two numbers in the second subset, and the next three numbers in the third subset and so on. Find in terms of a formula for the first member of the subset.

NB: *Start with number of whole number(s) in the first partition, second partition, etc*.

# Number and Algebra related problems: SHS Level

**Solution**

| Subset (n) | Elements in subset | First term ($F_n$) | Approach 1: Pattern between subset and first term | First difference | Second difference |
|---|---|---|---|---|---|
| 1 | {0} | 0 | $\frac{1 \times 0}{2}$ =0 | | |
| 2 | {1, 2} | 1 | $\frac{2 \times 1}{2} = 1$ | 1 | |
| 3 | {3, 4, 5} | 3 | $\frac{3 \times 2}{2} = 3$ | 2 | 1 |

# Number and Algebra related problems: SHS Level

| | | | | | |
|---|---|---|---|---|---|
| 4 | {6, 7, 8, 9} | 6 | $\frac{4 x 3}{2} = 6$ | 3 | 1 |
| … | .. | … | … | … | … |
| N | | | $\frac{n(n-1)}{2} = F_n$ | $n-1$ | |

# Geometry-Related Teasers

- Generally, it is a good thing to begin lessons with tasks that immediately grab students’ attention while serving a mathematical purpose.
- Since these tasks serve as ‘reawakening’ students before the main lesson begins it is expected that they take not more than five minutes, easy to do, exciting, and also mathematically worthwhile – involves a mathematical idea, principle, procedure, etc. that students get to learn.

# Geometry-Related Teasers

- Such tasks are referred to as warm up tasks or teasers.

**Activity 1:** Squares and Cubes

- From your content courses you have come across this relation $^{2}+\ ^{2}=\ ^{2}$ also known as the Pythagoras triples.
- Now check for pairs whose squares add up to a cube number e.g.

# Geometry-Related Teasers

$47^2 + 52^2 = 17^2$. Your task is to use your calculator to investigate other sets of numbers *a, b, c* that satisfy the relation ${}^2 + {}^2 = {}^3$. For example, the following are some sets with all the numbers a, b, and c being less than 50:

$2^2 + 2^2 = 2^3$ $\quad$ $5^2 + 10^2 = 5^3$

$16^2 + 16^2 = 8^3$ $\quad$ $2^2 + 11^2 = 5^3$

- Your turn: Can you find more sets of numbers satisfying this condition?
- Find also those of them having the property ${}^3 + {}^3 = {}^2$

e.g. $2^3 + 2^3 = 4^2$.

# Geometry-Related Teasers

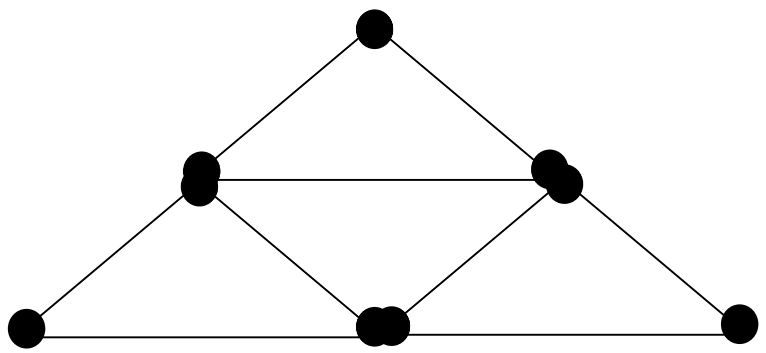
What pattern have you observed and what is the generalization?

**Activity 2:** Matchsticks Problems:

a)Matchstick Triangle

- Nine matchsticks are arranged to form four small equilateral triangles as shown.
- Now find a way of arranging only six of the matches to form four triangles of the same shape and size

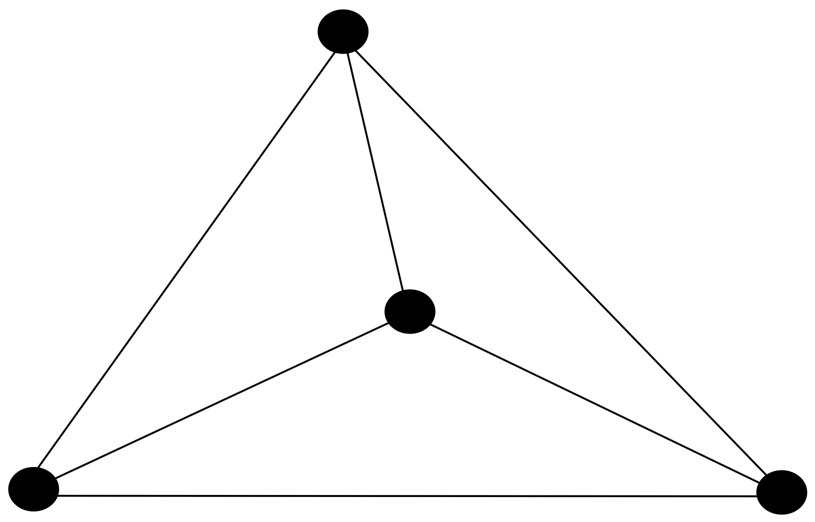
# Geometry-Related Teasers



**Solution:**

- The result should be a tetrahedron, a triangular pyramid with 4 triangular faces as shown.

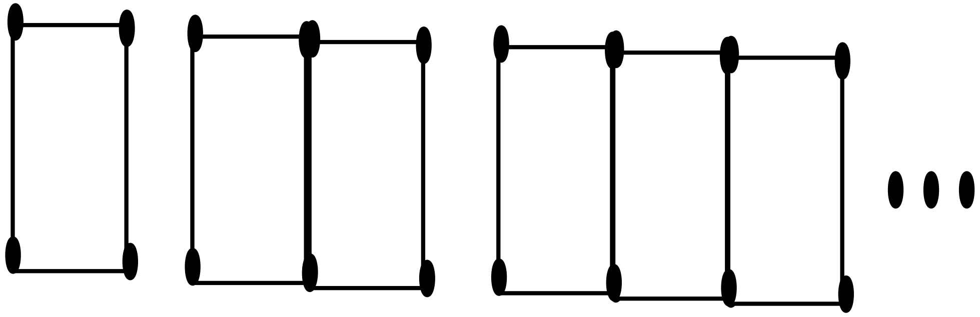
# Geometry-Related Teasers



# Geometry-Related Teasers

**b)Matchsticks squares 1**

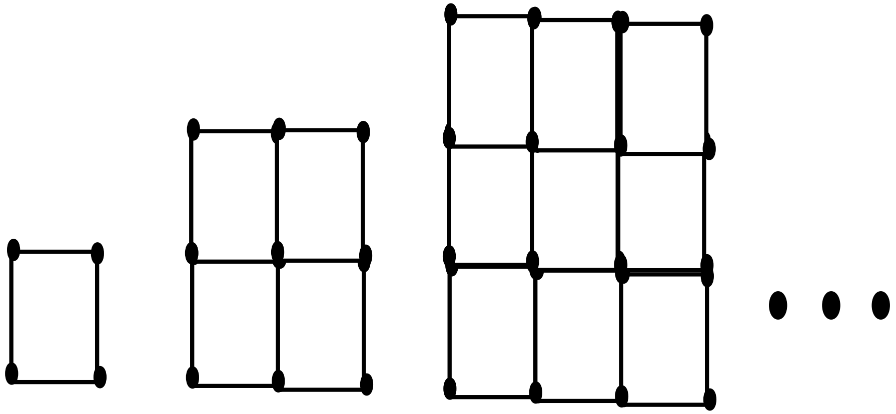
- How many matchsticks are needed to make 13 squares in a row, the side of each square being the length of a match, as shown in the following sequence?



# Geometry-Related Teasers

**c)Matchsticks squares 2**

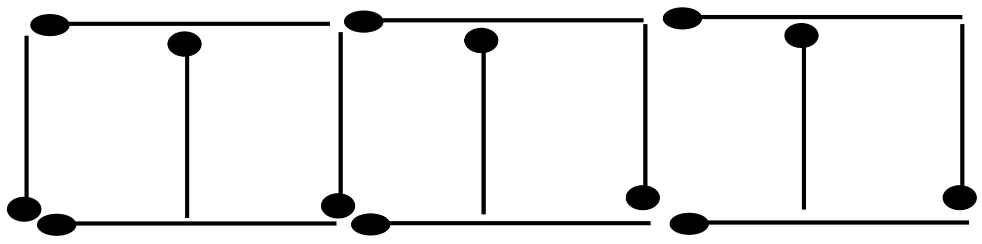
- How many matchsticks are needed to make N2 unit squares in a square array as shown in the following sequence?



# Geometry-Related Teasers

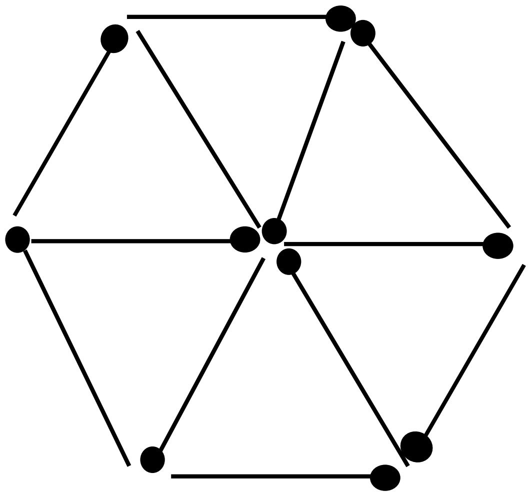
**The farmer's sheep-pens**

- The drawing shows how a farmer used thirteen identical woods to make six identical sheep pens. Unfortunately one of the woods was damaged. Use twelve woods and show how the farmer can still make six identical pens.



**Solution**

The solution possible is an unusual one which looks as shown.



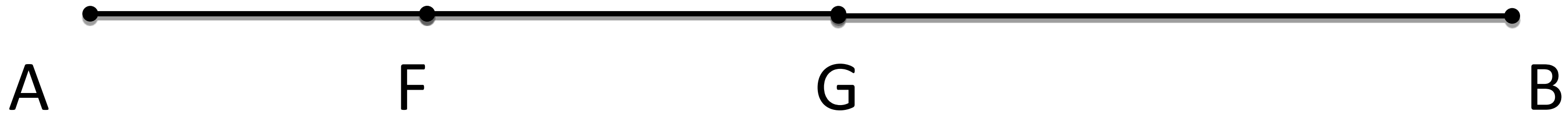
# Geometry-Related Problems – Primary Level

- Geometry, according to Masingila, Lester and Raymond (2011) is among the richest and oldest branches of mathematics and that Geometry is about the study of space experiences.
- The real objects we see around us are solid geometric shapes (space).
- On the faces of these solid objects are flat (plane) shapes.
- Out of space and plane shapes we study about concepts such as points, lines, angles, lengths, areas and volumes.

# Geometry-Related Problems – Primary Level

**Activity 1: Line Segments**

a) List all the line segments on the figure shown below:




b) Investigate the number line segments on a line with 10 points on it including the end points.

# Geometry-Related Problems – Primary Level

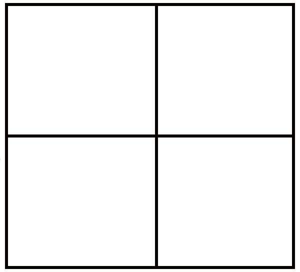
**Solution**

a) $\overline{AF}, \overline{AG}, \overline{FG}, \overline{FB}, \overline{GB}, \overline{AB}.$

b) 45 line segment.

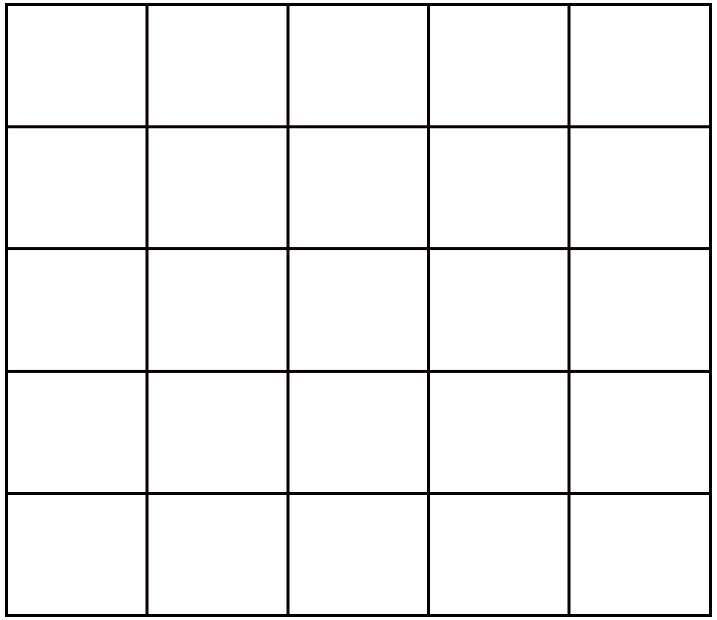
# Geometry-Related Problems – Primary Level

**Activity 2:** Squares and Rectangles



a) Investigate the number of squares and rectangles shown in the figure below.

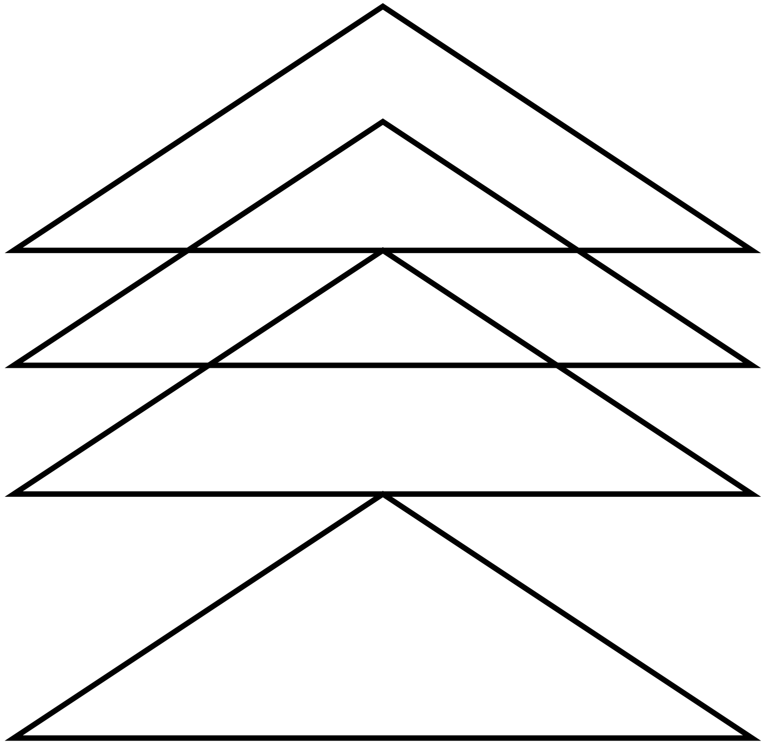
b) Investigate the number of squares for the figure below:



# Geometry-Related Problems – JHS Level

**Activity 1:** Christmas Tree

How many triangles and quadrilaterals can you see?



**Solution**

There are 6 triangles and 4 quadrilaterals.

**Activity 2:** Areas and Perimeters

# Geometry-Related Problems – JHS Level

A farmer has 12 logs to make a border around a field. Each log is I m long. The field must be rectangular.

1. (i) What is the biggest area of field the farmer can make?

   (ii) What is the smallest area of field the farmer can make?
2. The farmer now has 14 logs. Each is 1 m long. What is the biggest and the smallest fields he can make?
3. Try for different number of logs, odd and even and draw a conclusion about maximum and minimum areas.

# Geometry-Related Problems – JHS Level

| Length | Width | Area |
|---|---|---|
| 1 | 5 | 5 |
| 2 | 4 | 8 |

# Geometry-Related Problems – JHS Level

**Activity 2:** Areas and Perimeters

**Solution**

| | | |
|---|---|---|
| 3 | 3 | 9 |

You may draw a table put in possible dimensions that make a perimeter of 12.

Notice from table that the maximum area is 9 squareunits and the minimum is 5 square units

Do same for 14 logs (perimeter 14 units)

# Geometry-Related Problems – JHS Level

**Activity 3:** Rectangles –perimeters and areas A rectangle has perimeter of 40 cm.

1. What might its area be?
2. What is the minimum area? What are the dimensions? Give answer to the nearest whole number?
3. What is the maximum area? What are the dimensions? Give answer to the nearest whole number.
4. Investigate with four different rectangles and their areas.

# Geometry-Related Problems – JHS Level

| Length | Width | Area |
|---|---|---|
| 1 | 19 | 19 |
| 2 | 18 | 36 |
| 3 | 17 | 51 |
| 4 | 16 | 64 |
| 5 | 15 | 75 |
| 6 | 14 | .. |

# Geometry-Related Problems – JHS Level

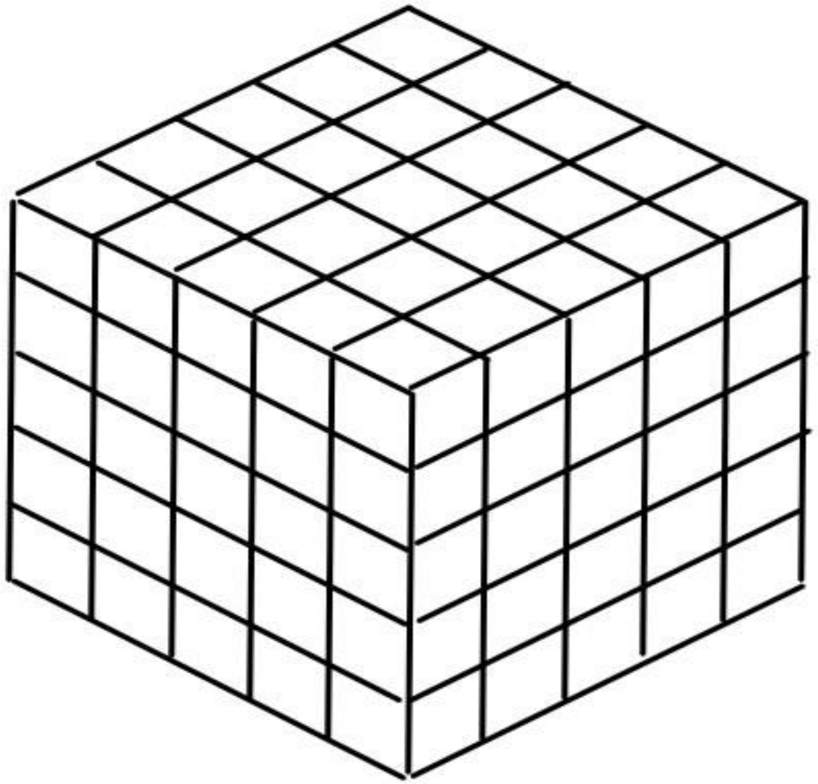
**Activity 3:** Rectangles –perimeters and areas **Solution**:

Notice the maximum area for a perimeter of 40 cm is 100 $cm^2$

| | | |
|---|---|---|
| … | … | … |
| 10 | 10 | 100 |

# Geometry-Related Problems – SHS Level

**Activity 2:** Painted Cube

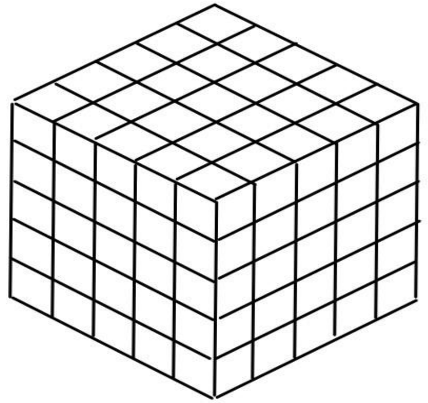


# Geometry-Related Problems – SHS Level

Imagine you have a 5 x 5 x 5 cube made up of unit cubes. The large cube is painted to cover all 6 sides.

**Activity 2:** Painted Cube

1. How many small cubes are painted on 3 faces? Where are these small cubes located on the larger cube?
2. How many small cubes are painted on 2 faces? Where are these small cubes located on the larger cube?



# Geometry-Related Problems – SHS Level

3.How many small cubes are painted on 1 face? Where are these small cubes located on the larger cube?

4.How many small cubes are with no painted faces? Where are these small cubes located on the larger cube?

| ***n sides*** | ***0 sides painted*** | ***1 side painted*** | ***2 sides painted*** | ***3 sides painted*** | ***cubes*** |
|---|---|---|---|---|---|
| 2 | 0 | 0 | 0 | 8 | 8 |
| 3 | 1 | 6 | 12 | 8 | 27 |

# Geometry-Related Problems – SHS Level

Solution

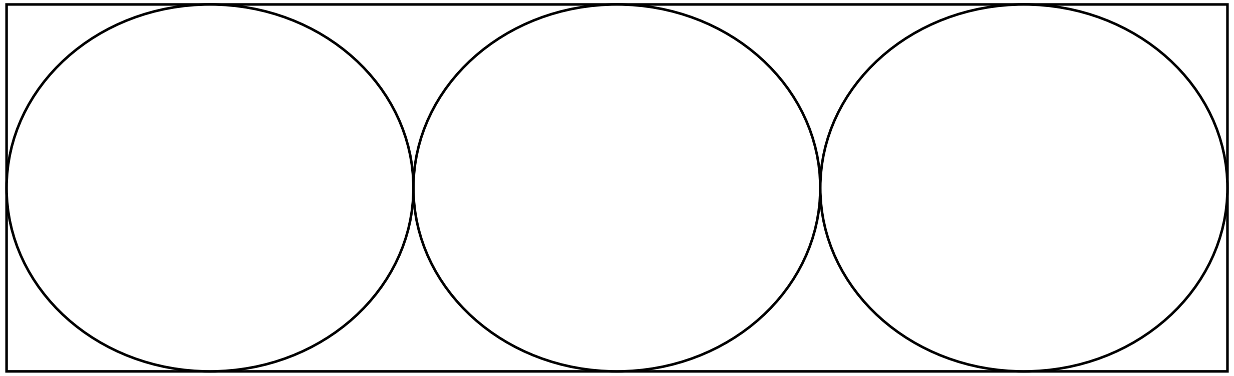
| | | | | | |
|---|---|---|---|---|---|
| 4 | 8 | 24 | 24 | 8 | 64 |
| 5 | 27 | 54 | 36 | 8 | 125 |
| 6 | 64 | 96 | 48 | 8 | 216 |
| 7 | 125 | 150 | 60 | 8 | 343 |
| 8 | 216 | 216 | 72 | 8 | 512 |
| 9 | 343 | 294 | 84 | 8 | 729 |
| 10 | 512 | 384 | 96 | 8 | 1000 |
| | $(n-2)^3$ | $6(n-2)^2$ | $(n-2)$ | 8 | $n^3$ |

$n$

# Estimation and Measurement

**Activity 1:** Inscribed Circles in a Rectangle.

- Three identical circles are inscribed in a rectangle as shown below. If the length of the rectangle is 90 cm, find the distance between the centres of the two end circles.



# Estimation and Measurement

**Solution**

- Since the three circles are identical, the three diameters of the circles will be the same. This means one diameter will be of length, 30 cm. The radius of each circle will therefore be 15 cm. There are 4 radii between the centers of the end circles. Hence the distance required will be 60 cm.

**Activity 2**

# Estimation and Measurement

A boy has three pieces of string of lengths, 7 cm, 3 cm, and 2 cm respectively. Investigate the various ways the boy can use these three pieces to cut strings of lengths:

a)6 cm  b) 8 cm  c) 13 cm

**Solution**

a)For 6 cm, the boy can use all the three strings to cut a length of 12 cm. The boy cans the fold the 12 cm string into two equal parts and the cut to obtain the 6 cm needed.

# Estimation and Measurement

Explore the solution to the others and discuss your solutions with your mates during F.T.F session.

# LOGIC – RELATED PROBLEM SOLVING ACTIVITIES

**Warm Up Activities**

- Warm up activities are generally used to set the stage for learning.
- The pupils’ readiness to learn depends on a teacher’s ability to capture the minds of the pupils, arouse their interest and sustain their interest throughout the learning process.
- These tasks are to ‘reawaken’ students before the main lesson begins.

# LOGIC – RELATED PROBLEM SOLVING ACTIVITIES

- They are to be very brief and stimulating, easy to do, and also mathematically worthwhile.

**Activity 1:** The Wolf, the Goat, and the Cabbage

You are travelling through a difficult country, taking with you a wolf, a goat, and a cabbage. All during the trip the wolf wants to eat the goat, and the goat wants to eat the cabbage, and you have to be careful to prevent either calamity. You come to a river and find a boat which can

# LOGIC – RELATED PROBLEM SOLVING ACTIVITIES

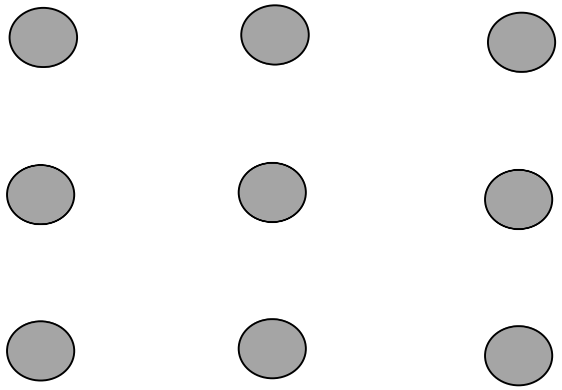
take you across, but it’s so small that you can take only one passenger at a time – either the wolf, or the goat, or the cabbage. You must never leave the wolf alone with the goat, nor the goat alone with the cabbage.

So how can you get them all across the river?

**Activity 2:** Nine dots

Nine dots in a square 3 by 3 array are to be joined by four consecutive line segments, without removing the pencil from paper or retracting any part of the path. Show how you can do this

# LOGIC – RELATED PROBLEM SOLVING ACTIVITIES



# Magic Square Activities

**Activity 1:** Arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 in the square such that any row, column and diagonal have the same sum (15)

| | | |
|---|---|---|
| | | |
| | | |
| | | |

# Magic Square Activities

**Solution**

| | | |
|---|---|---|
| 2 | 9 | 4 |
| 7 | 5 | 3 |
| 6 | 1 | 8 |

# Magic Square Activities

**Activity 2**

Make four 4 by 4 magic squares using the numbers 1, 2, 3, 4, . . . 16. Here is one example.

| | | | |
|---|---|---|---|
| 16 | 3 | 2 | 13 |
| 5 | 10 | 11 | 8 |
| 9 | 6 | 7 | 12 |
| 4 | 15 | 14 | 1 |

# Magic Square Activities

| | | | |
|---|---|---|---|
| | | 11 | |
| 14 | | | |
| | | | |
| | | | 16 |

Can you construct different versions?

# Magic Square Activities

| | | | | |
|---|---|---|---|---|
| 17 | 24 | 1 | 8 | 15 |
| 23 | 5 | 7 | 14 | 16 |
| 4 | 6 | 13 | 20 | 22 |
| 10 | 12 | 19 | 21 | 3 |

# Magic Square Activities

**Activity 3:**

| | | | | |
|---|---|---|---|---|
| 11 | 18 | 25 | 2 | 9 |

- Arrange the numbers 1, 2, 3, 4, 5, 6, -, -, -, 25 in the square such that any row, column and diagonal have the same sum (65)

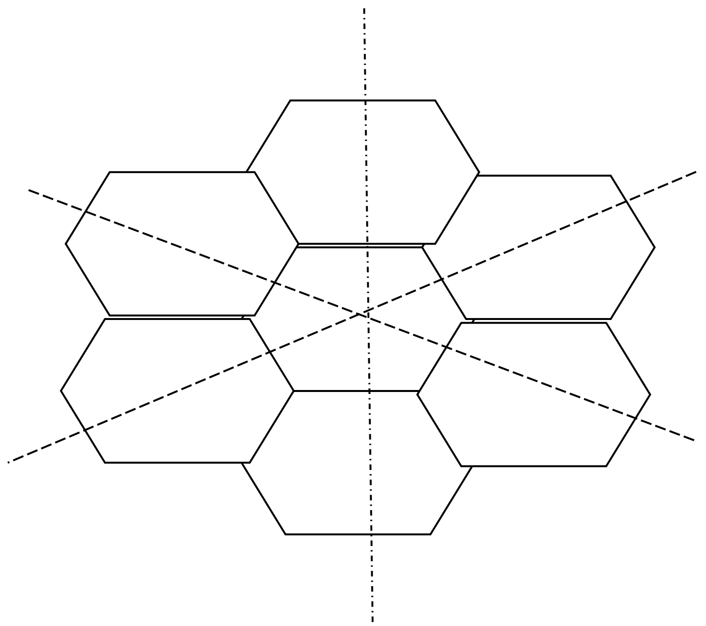
**Solution**

# Problem Solving Activities on General Tasks

**Activity 1:** Hexagon Sum

- Write the numbers 1 through 7, one in each hexagon, so that all three lines across the middle add up to a total of 12.

# Problem Solving Activities on General Tasks



**Activity 2:**

# Problem Solving Activities on General Tasks

Ten coins are arranged in a line as shown. A move consists of picking one coin, leaping over two coins and landing on another coin. Show that it is possible to arrange the coins into five pairs equally spaced.
Describe the process clearly.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

**Activity 3:** Leaping frogs and toads

# Problem Solving Activities on General Tasks

Across a stream runs a row of eight stepping stones. On one side of the stream, on the first five stones, sit five Frogs and they want to get across to the other side. There is one empty stone in the middle. On the other side are three Toads waiting to come across the other way.

Only one frog moves at a time. Any frog may hop over one toad or a toad may hop over one frog onto an empty stone.

Can you get all the animals across the river, and what is the smallest number of leaps?

# Problem Solving Activities on General Tasks

Any rule to guide?

**Activity 4:** Amule - Who is married to whom?

- Three sisters, Amule, Beauty and Confidence engaged in the jobs of architecture, baker and caterer. They got married to Mr. Abu, Dr. Bubune and Mr. Cole. None of the first letters in their first names and last names or occupations match up – so Mr. Abu is NOT married to Amule, and she is not the architect. If Cole's wife is NOT a baker, who is married to whom? Indicate the occupation of each lady.

# Problem Solving Activities on Logic

**Activity 1:** Dots on dice

1) The dots on the opposite faces of a dice add up to 7.

a) Imagine rolling one dice. The score is the total number of dots you can see. You score 17. Which number is face down?

b) Imagine rolling two dice. The dice do not touch each other. The score is the total number of dots you can see. Which numbers are face down to score 30?

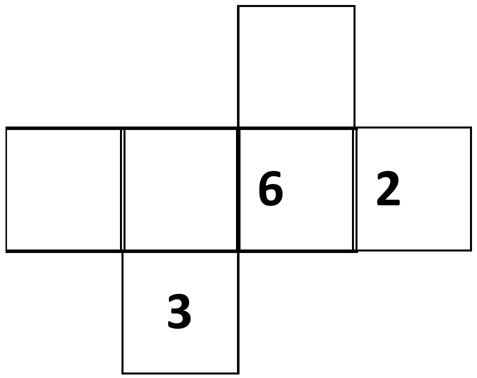
# Problem Solving Activities on Logic

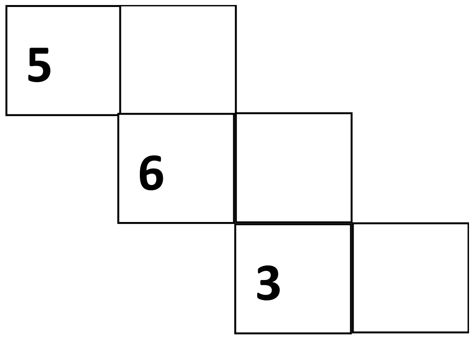
**Activity 2:**

Each of the two shapes shown can be folded up to make a dice. In each case three of the numbers are missing.

Show how to make the squares correctly so that the numbers on the opposite faces of the cube add up to 7.

# Problem Solving Activities on Logic




**Activity 3:**

# Problem Solving Activities on Logic

A spider is trying to climb a wall that is 15 metres high. In each hour, it climbs up 3 metres, but falls back 2metres. In how many hours will it reach the top of the wall? Explain your answer.

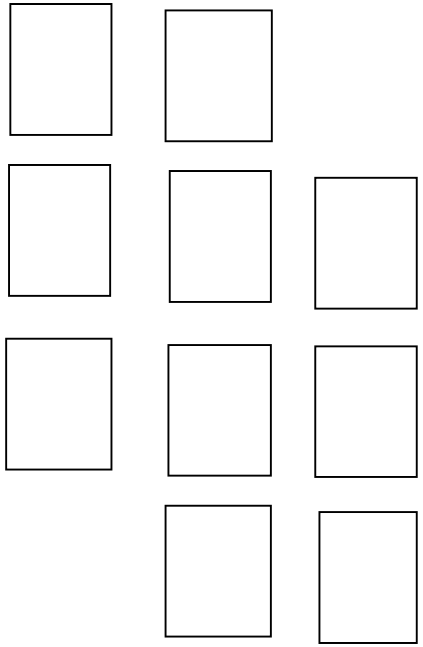
**Answer** is 13 hours. Why? (Share with your mates during F.T.F.)

# Problem Solving Activities on Logic

**Activity 4:** Mathematics Games Take ten cards numbered 0 to 9. Arrange the cards to look like the diagram shown.

Do it so that no two consecutive numbers are next to each other, horizontally, vertically and diagonally.

Find all the possible ways you can do it.



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