Module for Post-Graduate Diploma in Education Programme

ESC711M:
METHODS OF TEACHING MATHEMATICS

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UNIT 1: THE SENIOR HIGH SCHOOL CORE MATHEMATICS SYLLABUS

Mathematics as a subject of study can be described as a creative industry. It is basically a human activity that ascends from practices and becomes an essential part of culture and society, of everyday life and work. One thing that is seemingly quite hard to attain is how to teach mathematics effectively. This is because sometimes students found mathematics a boring subject, nevertheless as a student-teacher, you must prepare yourself squarely enough to face and address these problems. You must be creative and flexible enough to achieve your set objectives and goals. Methods are procedures that one follows to accomplish objectives, it stands for a specified course that serves as a guide in order "not to get lost on the way". As a student-teacher and a mathematics teacher someday, you must have lots of methods known in teaching mathematics in order to be more effective and creative in the process of students learning your subject. Before we can be effective in teaching mathematics, we need to have good knowledge about what we are supposed to be teaching as prospective mathematics teachers and how students learn mathematics. We are familiar with why we teach mathematics at basic and high schools.

Learning outcome(s)
By the end of the unit, the participant will be able to:

- identify and explain some rationale for studying mathematics;
- explain the general aims of the core mathematics teaching syllabus;
- identify and explain the general objectives of the core mathematics syllabus;
- identify and explain the scope of the mathematics syllabus;
- identify and explain the structure and organization of the mathematics syllabus; and
- familiarize yourself with an extract of the structure of the syllabus.

SESSION 1: THE STUDY OF THE SENIOR HIGH SCHOOL CORE MATHEMATICS SYLLABUS

The Cockcroft Report on "Why teach mathematics?" notes that mathematics is useful for everyday life, science, commerce, and for industry. It provides a powerful, concise, and unambiguous means of communication and also a means to explain and predict. Mathematics attains its power through its symbols, which have their own 'grammar' and syntax. It also develops logical thinking and has aesthetic appeal. It is believed that those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence has the potential for opening doors to productive futures in this era of technological advancement. These doors are closed to those who are perceived to lack this mathematical competence. Every student must therefore be given the opportunity and support necessary to learn significant mathematics with depth and understanding. A true understanding of mathematics may now be considered a prerequisite for "intelligent citizenship" (NCTM, 2000). This session deals with the rationale for the SHS Core Mathematics syllabus and introduces you to the general aims and objectives of the core mathematics syllabus and the various areas of mathematics content identified to be taught and learnt at SHS.

Learning outcomes
By the end of the session, the participant will be able to;

- explain the main rationale for teaching mathematics at the senior high school level.
- identify several aims for teaching core mathematics at the Senior High School.
- identify the general objectives of the Core mathematics syllabus.
- identify the scope of the Core mathematics syllabus.

RATIONALE FOR SENIOR HIGH SCHOOL CORE MATHEMATICS SYLLABUS

Effective knowledge of science and mathematics is crucial to an individual's development in almost all areas of life. For this reason, the education systems of many countries, including Ghana, put a great deal of emphasis on the study of mathematics. The main **rationale** for the SHS Core mathematics syllabus is focused on attaining one crucial goal: to enable all Ghanaian young persons to acquire the mathematical skills, insights, attitudes, and values that they will need to be successful in their chosen careers and daily lives. The Core Mathematics syllabus is based on the premise that all students can learn mathematics and that all need to learn mathematics. This means that mathematics teachers should not brand any student as incapable of learning mathematics in the SHS syllabus. Every student has the ability to learn the mathematics they are supposed to learn at the SHS. Only that some will be very fast in grasping the concepts, and others may require more time but given the right mathematics environment they can achieve a reasonable amount of the required knowledge. All SHS students need to learn mathematics because they all need to be functional in society. Denying them the opportunity to learn mathematics because they show some mathematics anxiety is equivalent to making them to be dysfunctional in life. Rather mathematics teachers are to make frantic efforts to support students develop positive attitude towards mathematics and be helped to learn mathematics with understanding.

The syllabus is therefore, designed to meet expected standards of mathematics in many parts of the world (MOE, 2010). The student is expected, at the SHS level, to develop the required mathematical competence to be able to use his/her knowledge in solving real life problems and secondly, be well equipped to enter into further study and associated vocations in mathematics, science, commerce, industry and a variety of other professions.

GENERAL AIMS FOR THE CORE MATHEMATICS SYLLABUS

The goals are developed to help attain the main rationale for teaching core mathematics at the SHS level. They are typically broad general statements that describe what the programme plans to accomplish. The general aims are broader means to attain the stated rationale of students acquiring the requisite mathematics concepts and skills that can make them functional in life. To meet the demands expressed in the rationale, the SHS Core Mathematics syllabus is designed to help the student to:

1. develop the skills of selecting and *applying* criteria for classification and generalization.
2. *communicate* effectively using mathematical terms, symbols and explanations through logical reasoning.
3. use *mathematics in daily life* by recognizing and applying appropriate mathematical *problem-solving* strategies.
4. understand the process of measurement and use appropriate measuring instruments.
5. develop the ability and willingness to perform *investigations* using various mathematical ideas and operations.
6. work *co-operatively* with other students in carrying out activities and projects in mathematics.
7. develop the values and *personal qualities* of diligence, perseverance, confidence, patriotism and tolerance through the study of mathematics
8. use the *calculator and the computer* for problem solving and investigations of real life situations
9. develop *interest* in studying mathematics to a higher level in preparation for professions and careers in science, technology, commerce, industry and a variety of work areas.

10. appreciate the *connection* among ideas within the subject itself and in other disciplines, especially Science, Technology, Economics and Commerce

GENERAL OBJECTIVES FOR THE CORE MATHEMATICS SYLLABUS

In order to achieve the stated general aims of the Core mathematics syllabus, a number of objectives have been identified and captioned as General objectives which are more linked to the instructional periods in the classroom. General objectives are therefore more specific than aims. Objectives serve as the building blocks or steps towards achieving a goal. Every instructional period needs to have a specified goal to attain. A number of units in the syllabus together is expected will contribute to bringing about a certain kind of change in the learners' behaviour. For example, a number of topics in the syllabus contribute to helping students to develop logical and abstract thinking or to develop interest in learning mathematics to a higher level. The general objectives include the following.

By the end of the instructional period students will be able to:

1. develop computational skills by using suitable methods to perform calculations;
2. recall, apply and interpret mathematical knowledge in the context of everyday situations;
3. develop the ability to translate word problems (story problems) into mathematical language and solve them with related mathematical knowledge;
4. organize, interpret and present information accurately in written, graphical and diagrammatic forms;
5. use mathematical and other instruments to measure and construct figures to an acceptable degree of accuracy;
6. develop precise, logical and abstract thinking;
7. analyze a problem, select a suitable strategy and apply an appropriate technique to obtain it's solution;
8. estimate, approximate and work to degrees of accuracy appropriate to the context;
9. organize and use spatial relationships in two or three dimensions, particularly in solving problems;
10. respond orally to questions about mathematics, discuss mathematics ideas and carry out mental computations;
11. carry out practical and investigational works and undertake extended pieces of work;
12. use the calculator to enhance understanding of numerical computation and solve real life problems

SCOPE OF THE CORE MATHEMATICS SYLLABUS

The syllabus is based on the notion that an appropriate mathematics curriculum results from a series of critical decisions about three inseparable, linked components: content, instruction and assessment. Consequently, the syllabus is designed to put great deal of emphases on the development and use of basic mathematical knowledge and skills. This session deals with the various areas of mathematics content identified to be taught and learnt at SHS. There are 30 topics to be handled in the three years of SHS. SHS1 and SHS 2 each has 13 topics to be treated. There are four topics for SHS3. The 30 topics cover seven major content areas.

The major areas of content covered in all the Senior High School classes are:

- Numbers and Numeration
- Plane Geometry

- Mensuration
- Algebra,
- Statistics and Probability
- Trigonometry
- Vectors and
- Transformation in a Plane.

Number and Numeration covers specific topics like Sets and Operations on set, Real number System, Surds, Number Bases, Ratio and rate and Percentages I all to be done in SHS1. In SHS2, we have modular arithmetic and Percentages II, and in SHS 3, we have Logical reasoning. ***Plane Geometry*** includes Plane Geometry I in SHS1, Plane Geometry II (Circle theorems) in SHS2 and Constructions in SHS3. Topics falling under Mensuration are Mensuration I in SHS2 and Mensuration II in SHS3. Under ***Algebra,*** we have Algebraic expressions, Relations and functions, Inequalities in SHS 1; Indices and Logarithm, Simultaneous Linear equations; Variations; Quadratic functions and Sequences and series in SHS2. ***Statistics and Probability*** is done in SHS1 as Statistics I and as Statistics II and Probability in SHS 2. ***Trigonometry*** is treated in SHS 2 and SHS 3 as Trigonometry I and Trigonometry II respectively. The last content area, Vectors and Transformation in a plane, is broken down as Bearing and Vectors in a plane and Rigid Motion I in SHS1 and as Rigid motion II and Enlargement in SHS2.

"Problem solving and application" has not been made a topic by itself in the syllabus but is expected to be an integral part of all content areas. Nearly all topics include solving word problems as activities. Teachers are to incorporate appropriate problems that will require mathematical thinking rather than mere recall and use of standard algorithms. Other aspects of the syllabus should provide opportunities for the students to work co-operatively in small groups to carry out activities and projects which may require out-of-school time. The level of difficulty of the content of the syllabus has been designed to be within the knowledge and ability range of Senior High School students.

Key ideas

- The main **rationale** for the SHS Core mathematics syllabus is focused on attaining one crucial goal: to enable all Ghanaian young persons to acquire the mathematical skills, insights, attitudes, and values that they will need to be successful in their chosen careers and daily lives.
- The goals are developed to help attain the main rationale for teaching core mathematics at the SHS level.
- Objectives serve as the building blocks or toward achieving a goal
- There are 30 topics to be handled in the three years of SHS.
    - SHS1 and SHS 2 each has 13 topics to be treated.
    - There are four topics for SHS3.

Reflection

1. Explain the main rationale for teaching mathematics at the SHS level.
2. State the premises on which the Core Mathematics syllabus is built.
3. State six general aims of teaching SHS Core Mathematics.
4. Identify and explain eight general objectives of the Core Mathematics syllabus.
5. Identify the main scope of the SHS Core Mathematics syllabus.

6. Identify the specific units of the SHS syllabus which fall under each of the following major content areas:

a. Algebra;

b. Geometry;

c. Number and Numeration.

SESSION 2: THE STRUCTURE OF THE SENIOR HIGH SCHOOL CORE MATHEMATICS SYLLABUS

The content areas of the syllabus and the breakdown into subtopics are presented in a table, referred to as the structure and organization of the syllabus. To go along with each of the subtopics are suggested specific objectives for teaching the content, suggested teacher and learner activities and the evaluation. This session discusses the contents of the structure and organization of the syllabus. It further gives an extract of the structure of the core mathematics syllabus.

Learning outcomes

By the end of the session, the participant will be able to:

- distinguish between unit and content;
- explain what syllabus reference number (SRN) is;
- explain teacher and learner activities suggested in the syllabus.
- identify teaching and learning activities aligned with specified contents and specific objectives

STRUCTURE AND ORGANIZATION OF THE CORE MATHEMATICS SYLLABUS

Each year's work has been divided into units with the unit topics for each year arranged in a suggested teaching sequence. This has not been done on term-by-term basis because it is difficult to predict accurately the rate of progress of students in each year. Teachers are advised not to force the instructional pace but rather ensure that students progressively acquire a good understanding and application of the material specified for each year's class work. It is hoped that no topics will be glossed over for lack of time because it is not desirable to create gaps in students' knowledge. There are five columns in the structure: Units, Specific Objectives, Content, Teaching and Learning Activities, and Evaluation.

Units

The first column headed **Units** specifies the major topics for the academic year. Two digits are used to indicate a particular topic, with the first digit showing the year while the second digit shows the sequential number of the unit. For example, Unit 1.1 refers to the first unit of SHS1, i.e., Unit 1 of Year One which is *Sets and Operations on Sets* as shown in Table 1. Unit 1.8 refers to the eighth topic of SHS1, which is *Word problems involving linear equations in one variable*. Similarly, Unit 2.10 is the tenth topic of SHS2, which is *Plane Geometry II(Circles)*; Unit 2.11 is the eleventh topic of SHS2 which is *Trigonometry I*, and Unit 3.1 is the first unit of SHS3, which is *Construction*.

Specific Objectives

The second column headed **Specific Objectives** shows the objectives for each unit. The specific objectives begin with numbers such as 1.1.1. These numbers are referred to as "Syllabus Reference Numbers (SRN)". The first digit refers to the year/class; the second digit refers to the unit, while the third refers to the rank order of the specific objective. For instance, 1.1.1 means SHS1, Unit 1 Specific Objective 1. That is, Specific Objective 1 of Unit 1 of SHS1. Unit 1.1 has four specific objectives from 1.1.1 to 1.1.4. Unit 1.8 has five specific objectives- 1.8.1 to 1.8.5.

Unit 2.10 has five specific objectives while Unit 2.11 has six objectives and Unit 3.1 has three objectives.

The **specific objectives** have been stated in terms of the students i.e. what the students will be able to do during and after instruction and learning in the unit. Each specific objective hence starts with the following "*The student will be able to…".* This in effect, means that you have to address the learning problems of each individual student. It means individualizing your instruction as much as possible such that the majority of students will be able to master the objectives of each unit of the syllabus. For instance,

SRN 1.1.1 is The student will be able to *determine and write the number of subsets in a set*.

SRN 1.8.4 is The student will be able to *solve word problems involving linear equations in one variable*.

SRN 2.10.2 is The student will be able to *state and use the circle theorems*.

SRN 2.11.1 is The student will be able to *define and compute the tangent, sine and cosine of an acute angle in degrees.*

SRN 3.1.2 is The student will be able to *construct a triangle or quadrilateral under given conditions*.

SRN provides an easy way for selecting objectives for test construction and an easy way for communication among teachers and other educators. The teacher would be able to use the syllabus reference numbers to sample objectives within units and within the year to be able to develop a test that accurately reflects the importance of the various skills taught in class.

Content

The third column is headed **Content.** It shows the mathematical skills, procedures, concepts, and rules required in the teaching and learning of the specific objectives. It is what we may call the core idea related to the unit (topic) you intend to teach or you want learners to acquire at the end of a topic or subtopic.

For example, the content that corresponds with SRN 1.1.1 is *Finding the number of subsets in a set with n elements*. For SRN 1.1.4, the content is *Three- set problems using Venn diagrams*. The content for SRN 1.8.4 is *Word problems involving linear equations in one variable*. The corresponding content for SRN 2.10.2 is *Circle theorems*. The content for SRN 2.11.1 is *Tangent, sine and cosine of acute angles*, while the content for SRN 3.1.2 is *Construction of triangles and quadrilaterals*.

In some cases, the content presented is quite exhaustive. It is expected that based upon your own experience (professional judgment), you should be able to narrow down what is the essential or core idea that learners should acquire and teach accordingly.

Teaching/Learning Activities (T/LA)

The fourth column headed **Teaching/Learning Activities (T/LA)** refers to activities that are likely to encourage maximum student participation in the lessons. The general aims of the mathematics can only be most effectively achieved when teachers create learning situations and provide guided opportunities for students to acquire as much knowledge and understanding of mathematics as possible. As such, the T/LA activities should be chosen and organized in such a way as to bring about the intended learning objectives. The T/LA therefore serve as the media through which students have opportunities to learn various mathematical skills, procedures, rules, and concepts thereby leading to the attainment of stated specific objective(s).

Students' questions are as important as teacher's questions. There are times when the teacher must show, demonstrate, and explain. But the major part of a students' learning experience should consist of opportunities for them to explore various mathematical situations in their environment so they can make their own observations and discoveries and record them.

Avoid rote learning and drill-oriented methods and rather emphasize participatory teaching and learning in your lessons. Teaching-by-telling does not lead to students' mathematical understanding. You are encouraged to re-order the suggested teaching/learning activities and also add to them where necessary in order to achieve optimum students' understanding of the topic being taught. Emphasize the cognitive, affective and psychomotor domains of knowledge in your instructional system wherever appropriate.

A suggestion that will help your students acquire the capacity for analytical thinking and the capacity for applying their knowledge to problems and issues is to begin each lesson with a practical and interesting problem. Select a practical mathematical problem for each lesson. The selection must be made such that students can use knowledge gained in the previous lesson and other types of information not specifically taught in class (MOE, 2010).

For example, the suggested T/LA for SRN 2.10.2 are:

- *Assist students to find the relationship between the angle subtended at the centre and that at the circumference by an arc.*
- *Guide students to find the value of the angle subtended by a diameter at the circumference.*
- *Guide students to find the relationship between opposite angles of a cyclic quadrilateral.*

The T/LA for SRN 3.1.2 include

Assist students to use a pair of compasses and ruler only to construct:

i. a triangle, given two sides and an included angle;
ii. a triangle, given two angles and a side;
iii. a quadrilateral under given conditions.

Evaluation

The fifth column headed **Evaluation,** shows suggestions and exercises for evaluating the lessons of each unit. Evaluation exercises can be in the form of oral questions, quizzes, class assignments, essays, project work, etc. Try to ask questions and set tasks and assignments, etc. that will challenge students to apply their knowledge to issues and problems that will engage them in developing solutions, and in developing observational and investigative skills as a result of having undergone instruction in this subject. The suggested evaluation tasks are not exhaustive. You are encouraged to develop other creative evaluation tasks to ensure that students have mastered the instruction and behaviours implied in the specific objectives of each unit.
For example, the evaluation for SRN 1.1.1 is, <u>Let students</u> *find the number of subsets in a given set*.

The evaluation for SRN 1.1.4 reads, <u>Let students:</u>

- *write or pose 2 set problems involving real life situations;*
- *solve 3 set problems involving real life situations.*

The evaluation for SRN 1.8.4 reads, Let students *solve word problems involving linear equations in one variable.*

The evaluation for SRN 2.10.2 reads, Let students *find missing angles using circle theorems.*

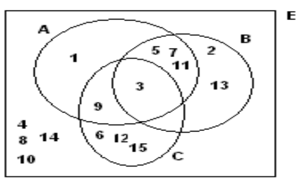
The evaluation for SRN 2.11.1 reads, Let students:

- *express the tangent, sine and cosine in relation to the sides of a given acute angle in a right-angled triangle*
- *read values of given trigonometric ratios of acute angles from tables and calculators*

Do not take the syllabus as a substitute for lesson plans. It is necessary that you develop a scheme of work and lesson plans for teaching the units.

EXTRACT OF THE STRUCTURE OF THE CORE MATHEMATICS SYLLABUS

Table 1: Extract of the Structure of the Core Mathematics Syllabus

UNIT	**SPECIFIC OBJECTIVES**	**CONTENT**	**TEACHING AND LEARNING ACTIVITIES**	**EVALUATION**
	The student will be able to:			Let students
Unit 1.1 Sets and Operations on Sets	1. determine and write the number of subsets in a set ……. ………. 4. find the solution to practical problems involving classifications using Venn diagrams	Finding the number of subsets in a set with *n* elements Three-set problems using Venn diagrams	Review with students description of sets; words/set builder notation; listing; Venn diagrams Guide students to deduce that the number of subsets in a set with *n* elements is given by 2^n. Review two-set problems. Guide students to solve problems involving three sets  i. List the elements of $(A\cap B)\cup C$ i. What is $n(A\cup B)\cap C$	find the number of subsets in a given set write or pose 2 set problems involving real life situations solve 3 set problems involving real life situations

…….	….	….		
Unit 1.8 Linear Equations and Inequalities	1.8.4: solve word problems involving linear equations in one variable	Word problems involving linear equations in one variable	Guide students to solve word problems involving linear equations in one variable.	solve word problems involving linear equations in one variable.
…….				
Unit 2.10 Plane Geometry II (Circle Theorem)	2.10.2: state and use the circle theorems	Circle Theorems	Assist students to find the relationship between the angle subtended at the centre and that at the circumference by an arc. Guide students to find the value of the angle subtended by a diameter at the circumference. Guide students to find the relationship between opposite angles of a cyclic quadrilateral.	find missing angles using circle theorems.
….	…..	…..	…..	
Unit 2.11 Trigonometry I	.11.1 define and compute the tangent, sine and cosine of an acute angle in degrees.	Tangent, sine and cosine of acute angles.	Guide students to use appropriate diagrams to define trigonometric ratios. E.g. $\tan\theta = \frac{AB}{OA} = \frac{A_1B_1}{OA_1} = \frac{A_2B_2}{OA_2} = \frac{A_3B_3}{OA_3}$	express the tangent, sine and cosine in relation to the sides of a given acute angle in a right-angled triangle read values of given trigonometric ratios of acute

			$\sin\theta = \frac{AB}{OB} = \frac{A_1B_1}{OB_1} = \frac{A_2B_2}{OB_2} = \frac{A_3B_3}{OB_3}$ $\cos\theta = \frac{OA}{OB} = \frac{OA_1}{OB_1} = \frac{OA_2}{OB_2} = \frac{OA_3}{OB_3}$ Guide students to read the values of given trigonometric ratios of acute angles from tables and calculators.	angles from tables and calculators
…….	……..	…….	……..	……..
Unit 3.1 Construction	construct 75°, 105°, 135° and 150° …. construct a triangle or quadrilateral under given conditions .3 construct a particular loci	Construction of 75° 105° 135° and 150°. Construction of Triangles and Quadrilaterals. Constructing loci	Review the construction of 30°, 45° , 60° and 90° with the students. Guide students to construct angles 75°, 105°, and 135°. Assist students to use a pair of compasses and ruler only to construct; . a triangle, given two sides and an included angle; . a triangle, given two angles and a side. . a quadrilateral under given conditions. Guide students to construct the locus of points equidistant from two or more fixed points and two or more intersecting straight lines	construct some given angles. construct triangles and quadrilaterals under given conditions solve loci related problems through construction

Key ideas

- Each year’s work has been divided into units with the unit topics for each year arranged in a suggested teaching sequence
- There are five columns in the structure: Units, Specific Objectives, Content, Teaching and Learning Activities, and Evaluation.
  - The first column headed **Units** specifies the major topics for the academic year.
  - The second column headed **Specific Objectives** shows the objectives for each unit
  - The third column is headed **Content.** It shows the mathematical skills, procedures, concepts, and rules required in the teaching and learning of specific objectives.
  - The fourth column headed **Teaching/Learning Activities (T/LA)** refers to activities that are likely to encourage maximum student participation in the lessons.
  - The fifth column headed **Evaluation,** shows suggestions and exercises for evaluating the lessons of each unit.

Reflection

1. Distinguish between unit and content in the core mathematics syllabus.
2. What does SRN 1.2.7 represent in the core mathematics syllabus?
3. Write down Teaching and Learning activities for teaching circle theorems
4. Write down one Teaching and Learning activity for teaching sets and operations on sets.
5. Identify specific objectives for teaching construction of loci.

UNIT 2: TEACHING METHODS 1

Teaching and learning are *two different, but related functions*, the process of teaching being carried out by one person (the teacher), while the process of learning is carried out by another (the student). For the teaching-learning process to work effectively there must be some connection or bridge between the teacher and the learner. Teacher needs special communication skills primarily involving talking in order to be effective in making these connections. Some aspects of teaching can be seen to encourage to a large extent what is termed Rote learning. This refers to the teaching of mathematical results, definitions and concepts by repetition and memorisation typically without meaning or supported by mathematical reasoning. A derisory term is *drill and kill*. In traditional education, rote learning is used to teach multiplication tables, definitions, formulas, and other aspects of mathematics. This unit covers what Effective teaching is, Characteristics of effective mathematics teacher, Definition of teaching method, Direct and indirect teaching methods, Lecture techniques, and Brainstorming and Role Play.

Learning outcome(s)

By the end of the unit, the participant will be able to:

- explain what effective teaching is;
- state and explain at least four characteristics of effective mathematics teacher;
- distinguish between Less-participatory and more-participatory teaching approaches;
- distinguish between direct and indirect teaching approaches;
- enumerate the advantages and disadvantages of lecture technique;
- explain brainstorming and role play teaching methods.

SESSION 1: EFFECTIVE TEACHING

Good teaching is conceived as so complex and creative that it defies analysis. The effective teacher is one who can demonstrate the ability to bring about intended learning goals, the two critical dimensions of effective teaching being <u>*intent*</u> and <u>*achievement*</u>. Without intent, the student's achievements become random and accidental rather than controlled and predictable. Without achievement of his intended learning goals, the teacher cannot truly be called effective.

Two of the most important features of teaching in the promotion of conceptual understanding are attending explicitly to concepts and allowing students to struggle with important mathematics. The ability of a teacher to achieve this is a sign of good teaching. This session deals with some features of effective teaching in general.

Learning outcomes

By the end of the session, the participant will be able to explain what is meant by effective teaching.

Teaching is a complex task yet educators agree on a list of characteristics of a good teacher. Ryan (1960) and his colleagues identified three main factors of effective teaching as follows:

- Teacher being warm and understanding (versus cold and aloof).
- Teacher being organized and business like (versus unplanned and slipshod).
- Teacher being stimulating and imaginative (versus dull and routine).

i. A warm and understanding teacher shows much love towards the students and the mathematics being taught. They are often times humorous in class and very approachable. They attempt to listen to their students and address relevant issues the students raise. On the other hand, teachers who are cold and aloof often create a certain distance between them and the students. Students do not feel comfortable getting close to them with their problems.

ii. Teachers who are organized and business like are often seen to be focused on academic activities in class. They are time conscious, and regular. Their activities in class are sequenced and straight to the task. Unplanned and slipshod teachers are often disorganized and lose focus.

iii. Stimulating and imaginative teachers have a lot of impact on their students. Their classes are usually lively and full of innovative ways of doing things. They make room for alternative approaches and keep improving teaching and learning. Student participation is often high.

On the other hand, many students feel bored in classes handled by teachers who are dull and routine.

Teachers rated nearer the positive poles of each factor are considered more effective than teachers rated nearer the negative poles. Rosenshine and Furst (1973) reviewed several studies and identified five characteristics consistently associated with gains in students' achievement. These are

1. Teacher is enthusiastic.
2. Teacher is business like and task-oriented.
3. Teacher is clear when presenting instructional content.
4. Teacher uses a variety of instructional materials and procedures.
5. Teacher provides opportunities for students to learn the instructional content.

A study conducted by Berliner and Ticker off (1976) on measuring teacher effectiveness in terms of students' gain on standardized achievement tests identified the following observable indications of effective teaching.

1. Students show knowledge and understanding, skills and attitudes intended by the curriculum as measured by performance on tests.
2. Students exhibit behaviour which indicates a positive attitude towards teachers and peers.
3. Students exhibit independent behaviour in learning curriculum content.
4. Students' behaviour indicates a positive attitude towards themselves as learners.
5. Students do not exhibit behaviour problems in class.
6. Students seem actively engaged in learning academically relevant material while the class is in session. This was described by Rosenshine and Berliner as ***academic engaged time*** to mean the amount of time they did spend on activities which involved the student in learning *academically relevant material*. The more the academic engaged time, the more the achievement.

To be effective B. O. Smith (1969) suggested four areas of knowledge a teacher must prepare in.

1. Command of theoretical knowledge about learning and human behaviour.
2. Display of attitudes that foster learning and genuine human relationships.
3. Command of knowledge in the subject-matter to be taught.

4. Control of technical skills of teaching that facilitate students' learning.

Reflection

1. What do you understand by the term *effective teaching*?
2. State and explain four observable evidence of an effective teaching of mathematics
3. State and explain three areas of knowledge that a mathematics teacher must be abreast of.

SESSION 2: CHARACTERISTICS OF AN EFFECTIVE MATHEMATICS TEACHER

Effective teaching describes a particular teacher who had been the most successful in helping the learner to learn. The characteristics of an effective teacher describe particular teacher's special personal qualities that learners felt had enabled the teacher to achieve success. An effective teacher must be happy with his /her job at all times. The teacher who every child in the school would love to have, the teacher children remember for the rest of their lives, do the job so that your students get the most out of what you are teaching them. Are you that teacher? Read on and learn more characteristics of an effective teacher of mathematics.

Learning outcomes

By the end of the session, the participant will be able to state and explain the characteristics of an effective mathematics teacher.

Note that each student is an individual with unique learning preference and needs. Students acquire knowledge, skills and attitudes at different times, rates and ways.

1. Students come to the classroom with preconceptions (or alternative conceptions) about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that are taught, or they may learn them for purposes of a test but revert to their alternative conceptions outside the classroom. This implies teachers must draw out and work with the preexisting understandings that their students bring with them. This can be in the form of diagnostic test prior to instruction.

2. To develop competence in an area of inquiry, students must: (a) have a deep foundation of factual knowledge, (b) understand facts and ideas in the context of a conceptual framework, and (c) organize knowledge in ways that facilitate retrieval and application. The implication is that teachers must teach some subject matter in depth, providing many examples in which the same concept is at work and providing a firm foundation of factual knowledge.

3. A "meta-cognitive" (i.e., thinking about one's own thinking) approach to instruction can help students learn to take control of their own learning by defining learning goals and monitoring their progress in achieving them. You should provide opportunities for students to reflect on their work so they can modify their own thinking. (National Research Council. In J. D. Bransford, A. L. Brown, 2000).

   The qualities of an effective teacher of mathematics are difficult to measure. He/she must first of all be an effective teacher as well as a competent instructor. He/she must be enthusiastic about mathematics and share this love with students. An effective mathematics teacher must:

i. have competency in the mathematics being taught as well as its prerequisites:- be competent and confident with the subject matter so that he can create positive mathematical experiences best suited for the development of the learner.

ii. have the desire to learn mathematics; learn how to educate the students; learn how to supply the optimum environment for each student; investigate new mathematical knowledge and effective teaching strategies. If you do not know an answer to a student's question, your thirst for knowledge should drive you to pursue the situation until you obtain that information. Stimulate your students to also make similar pursuits as you travel the road to new knowledge together, each at your own respective level and pace.

iii. be devoted to the profession – create a stimulating atmosphere conducive to learning. He must possess the desire, passion and patience to facilitate the learning of others. He must be in a position to help erase the fear and anxiety that mathematics represents to so many students. The effective mathematics teacher must be able to motivate all students to learn mathematics, perceive patterns and solve unconventional problems.

iv. be able to provide effective learning environment and incorporate real-life mathematical activities. The effective mathematics teacher must encourage students to learn by making connections through a variety of experiences. Knowledge is gained through the context of meaningful activities. He/she must emphasize student-centered instruction –constructivist approach to teaching mathematics which relies on the premise that knowledge is constructed by the learners as they attempt to make sense of their own experiences. The effective mathematics teacher must make students become active participants in the total learning environment.

v. be excited about learning and doing mathematics otherwise you cannot expect your students to be so. Your attitude and enthusiasm count greatly.

vi. believe that all students can learn mathematics and so develop meaningful activities for them. You must understand the nature of mathematics.

Reflection

1. A "meta-cognitive" approach to instruction can help students learn to take control of their own learning. Explain.
2. State and explain three characteristics of an effective mathematics teacher.
3. State and explain four things that an effective mathematics teacher should do in order to enhance effective teaching.
4. Enumerate three things that an effective teacher must NOT do in the mathematics classroom.

SESSION 3: LESS PARTICIPATORY AND MORE PARTICIPATORY TEACHING METHODS

Method deals with "how". How to enable the child learn is the final step of the execution of what we plan to teach in mathematics. A teaching method comprises the principles and activities/processes used for instruction to be implemented by teachers to achieve the desired learning by students. These strategies are determined partly on subject matter to be taught and partly by the nature of the learner. For a particular teaching method to be appropriate and effective it has to be in relation with the characteristic of the learner and the type of learning it is supposed to bring about.

Learning outcomes

By the end of the session, the participant will be able to explain with examples what is meant by 'less participatory and more participatory teaching methods'.

The approaches for teaching can be broadly classified into **less participatory pedagogy (teacher-centered) and more participatory pedagogy (student-centered)**. In **less participatory pedagogy** to learning, teachers are the main authority figure in this model. Students are generally viewed as "empty vessels" or lacking the capacity to make meaning on their own. Their primary role is to passively receive information (via lectures and direct instruction) with an end goal of testing and assessment. It is the primary role of teachers to pass knowledge and information onto their students. In this model, teaching and assessment are viewed as two separate entities. Student learning is measured through performance on tests and other forms of assessment techniques that may be determined by the teacher. This approach is also referred to as less *participatory because students are not actively involved in the teaching-learning process in terms of making conjectures, verifying whether an answer is correct or not, and do not share in the meaning making process.*. In this kind of instructional approach, students are reported to experience mathematics as "something 'done to them' rather than 'done by them'" (Solomon, 2007, p.90). The effect is that students tend to feel bored in class, come to view mathematics as a subject that has no meaning (.i.e. requires memorization), and find the classroom environment threatening to them. Less-participatory methods are lecture and teacher demonstrations.

In **more participatory pedagogy** or **student-centered** approach to learning, teachers and students play an active role in the learning process. In this case, the teacher does not serve as the sole repository of all knowledge; rather, students' contributions are elicited and valued as critical. The teacher's primary role is to coach (guide) and facilitate student learning and overall comprehension of material. Student learning is measured through both formal and informal forms of assessment, including group projects, student portfolios, and class participation. Teaching and assessments are connected; student learning is continuously measured during teacher instruction.

More participatory teaching methods stress practical activities that elicit from students their thinking, encourage invention of solution strategies, etc, and keep the learner motivated and more involved in the learning process. It also includes rich tasks which allow students to solve them using multiple approaches so that there can be meaningful conversations about whether a technique is appropriate or not and what counts as a mathematically appropriate response (Cobb, Gresalfi, & Hodge, 2009). Participatory methods include: Inquiry approach (investigation, project, discovery, brainstorming, discussion (group/class), games, field trip, exhibitions, role play.

Every teaching method has its unique procedure, advantage and limitations. Generally, several methods, approaches and techniques can be used in the process of teaching. Often, a teacher is faced with the challenge of determining the most appropriate methods for specific learning situations. Teachers must use methods that can assist learners to develop **critical thinking** skills and develop desirable attitudes and behavior. Activity centered methods are preferable, however, the adoption of this approach does not mean the teacher is never in control over the instructional process.

Reflection

1. What is a teaching method? 2. Distinguish between student-centered and teacher-centered methods of teaching 3. A mathematics teacher must employ methods that can help students develop critical thinking. Explain.

SESSION 4: DIRECT AND INDIRECT TEACHING METHODS

Different methods of teaching have been proposed or propounded by different educational thinkers or schools of thought in education. The direct method of teaching is usually used to teach a specific skill. It is a teacher-directed method, meaning that the teacher stands in front of the classroom and present the information. The indirect method on the other hand requires the teacher to take a facilitator role in guiding students. It is a student-led learning process in which the lesson doesn't come directly from the teacher. In the indirect teaching method, the key is to have students actively engaged in the learning process. This session discusses another classification of teaching methods as Direct and Indirect approaches.

Learning outcomes

By the end of the session, the participant will be able to distinguish between direct and indirect methods of teaching.

Flander's (1970) and his associates observed **two** contrasting styles of teaching – Direct and Indirect.

Direct Teaching Method

Direct teaching is the instance where the teacher places the highest priority on the assignment and completion of academic activities. Teacher selects and directs the learning task for the students. The aim is to maximize student learning time. It is characterized by teacher reliance on lecture, criticism, justification of authority and the giving of directions. Studies have shown that a strong academic focus manifests greater student academic engagement and achievement. A common form of direct instruction is the lecture mode. Lectures can make a positive contribution if they are dynamic, well planned and organized. The average teacher does 70% of the talking in the primary and secondary classrooms (Flanders, 1970; Perrott, 1977). Much of this time is spent in presenting new concepts and information to students using narration, description and explanation i.e. Lecture-explanation method – a teacher-centred technique in which interaction between teacher and students is minimal.

Indirect Teaching Method

Indirect teaching is characterized by teacher reliance on asking questions, accepting students' varied viewpoints and emotional needs, acknowledging students' ideas and building on it, and giving praise and encouragement. It promotes an increase in critical thinking and problem solving skills. Students are encouraged to explore and discover information through the use of open-ended questions, group discussions and activities, experiments and hands-on exercises. Studies have shown that students of indirect teachers learn more and have better attitudes toward learning than students of direct teachers.

However, *direct teaching* is effective and convenient for the teacher but may not be supportive of learning of most students.

Reflection

1. Distinguish between direct and indirect methods of teaching mathematics.
2. State and explain three merits and three demerits of indirect teaching method.
3. Explain the characteristics of each of the following:

a. Indirect teaching method;
b. Direct teaching method.

SESSION 5: LECTURE AND EXPLANATION TECHNIQUE

A lecture is a method of imparting information through a speech. Lecture is another name for a speech. In other words, a **lecture** is an oral presentation intended to present information or to teach people about a particular subject. Lectures are used to convey critical information, history, background, theories, and equations. Usually, the lecturer will be at the front of the room and recite information relevant to the content. Bligh (2000), argues that lectures "represent a conception of education in which teachers who know give knowledge to students who do not and are therefore supposed to have nothing worth contributing." In this session, we will discuss the characteristics of the lecture method in teaching mathematics.

Learning outcomes

By the end of the session, the participant will be able to;

- explain the features of lecture technique of teaching
- explain the merits and three demerits of the lecture method of teaching.

The lecture method is convenient and usually makes the most sense, especially with larger classroom sizes. This seems to make lecturing the standard for most courses. The teacher addresses all the students once, while still conveying the information that he or she feels is most important, according to the lesson plan. The lecture method facilitates large-class communication, but lecturers need to constantly and consciously be aware of student problems and encourage them to give verbal feedback. It can be used to arouse interest in a subject provided the teacher has effective writing and speaking skills.

Traditionally, teachers use lecture heavily in mathematics instruction. In many mathematics classrooms teachers teach new mathematics topics by presenting their students with lectures in which they outline the history of the mathematics topic and the steps necessary to complete mathematics processes. The teacher presents information and ideas – introduce topics, summarises the main points, and stimulate further learning.

Lecture-explanation, without any student participation should not usually exceed 10 – 20 percent of the lesson time. Asking students questions during the lesson is an example of a technique designed to create student involvement. Modern lectures generally incorporate additional activities, e.g. writing on a chalk-board, exercises, class questions and discussions, or student presentations.

Procedure: In using the lecture method of teaching;

a. The teacher speaks and the students listen.
b. The teacher gives ideas and the learner takes them.

c. The lecture method takes the form of a "one man show" where the learner remains passive.

The following are some ways of increasing the effectiveness of lecture method.

i. State instructional objectives clearly – both teacher and students must know what they want to accomplish at all times, what is it the student is expected to be able to do at the end of instruction?
ii. Learning activities must match the desired objectives. Inappropriate activities waste the time of teacher and students, cause confusion, and result in a lack of purpose and direction.
iii. Learning activities must be appropriate for each learner. Learner must have a reasonably good chance of being successful. Curriculum must be success-oriented.
iv. Teacher must do everything possible to ensure maximizing 'time on task' – the time students are engaged in active learning during a class period.
v. Teacher should maintain an academic focus during learning activities.
vi. Teacher should provide academic feedback of students' work.
vii. Teacher should handle discipline problems promptly.

Merits and Demerits of Lecture Methods

Lectures are a quick, cheap, and efficient way of introducing large numbers of students to a particular topic. It gives a quick exposure to new material, greater teacher control in the classroom, and facilitates large-class communication. Lecturing also permits the dissemination of unpublished or not readily available material.

The lecture method is criticized as a one-way method of communication that does not involve significant student participation. It places students in a passive (rather than an active) role. It requires the teacher to possess effective speaking skills.

Merits

1. The method is suitable when the class size is very large. When the number of students in a class is very large, this method is the only way out. All students are provided with equal opportunity to listen and learn.
2. When heavy syllabus is to be covered/ completed in a short time, lecture method is suitable.
3. The situation becomes impressive. When the teacher is delivering his lecture fluently, the students are listening attentively and there is a pin drop silence among them.
4. The method is convenient for the teacher.
5. The teacher and the learner feel satisfied at their respective places. Such satisfaction may be denounced as false but it is there.

Demerits

1. The method gives a false sense of satisfaction, which is dangerous and harmful.
2. The students remain mostly passive in the learning process
3. The students may remain inattentive during the lecture.
4. Experimentation is completely neglected. Students have no opportunity to discover facts for themselves.
5. It encourages rote learning, as students end up memorizing bits of information without understanding them.

Reflection

1. Explain the lecture method as a technique for teaching mathematics.
2. Describe how you will use the lecture method to teach a named mathematics topic
3. Distinguish between lecture and explanation method of teaching.
4. Identify five ways the mathematics teacher can make effective use of lecture method in teaching.
5. State and explain three merits and three demerits of lecture method.
6. Explain why you would advocate for the use of the lecture method in teaching mathematics.

SESSION 6: BRAINSTORMING AND ROLE PLAY

There are some teaching methods that encourage the use of discussion in class. A discussion approach consists of questions, answers and comments by both teacher and students. It involves feedback and student participation and so it is expected to be an effective method of learning. (McKeachie 1963; Abercrombie 1971). It is a useful preliminary or follow-up to an independent learning and is useful in helping students to work out complicated problems. This session deals with two other teaching methods called brainstorming and role-play.

Learning outcomes

By the end of the session, the participant will be able to explain the steps involved in the use of brainstorming and role-play to teach mathematics.

Brainstorming

It is a problem-solving technique used to generate ideas and encourage learner participation. Remember "two heads are better than one" so group thinking can enhance attainment of knowledge much faster and quite easily.

Procedure

1. Identify the topic to the group. (Be clear).
2. Set and enforce a time limit for the exercise (10 minutes maximum).
3. Focus on main issues/problem. Be specific, set relatively easy but challenging task.
4. Promote relaxed and cooperative atmosphere.
5. Challenge everyone to participate/contribute freely, building on others' ideas.
6. Allow everyone to air his/her views (no answer is right or wrong).

Stop brainstorming at the end of set time and allow discussion of ideas. Categorize all ideas for the groups to critically evaluate them for possible application.

Example 1: **Unseen Socks:** Ben's favourite colours are blue and green, so it's not surprising that he has six blue socks and six green socks in his sock drawer. Unfortunately, they are hopelessly mixed up and one day, in complete darkness, he has to grab some socks to wear. How many socks does he have to take from the drawer to make sure he gets a matching pair-either green or blue? (For some strange reason, his mother insists that his socks have to match!)

Example 2: A snail is at the bottom of a 50 metres deep well. Every day it crawls up 3 metres before slipping back 2 metres every night. How many days will it take it to escape from the well?

Role Play
It is an unrehearsed, informal dramatization in which learners spontaneously act out roles of a given character in a given situation e.g. Corner shops in a class for buying/selling. Scenario in class-story problem put in a scene. (Ratio concept).

Advantages

1. Useful to maintain and arouse interest.
2. Useful to provide a strong basis for discussion.
3. Useful to increase learners' understanding of themselves and others and to provide the opportunity for behaviour change.

Procedure

1. Describe background and setting of role play.
2. Select and brief actors and observers.
3. Have learners act the roles assigned to them. Have them portray roles as they believe the character would behave in that situation.
4. Have observers take notes/comments to be shared with group
5. Watch the learners to check boredom or interest.
6. Stop role-play when you feel the actors have shown the feelings and ideas which are important in the platform situation.

Initiate discussion after the Role-play to ensure that students get the intended message.

Reflection

1. Explain how you would use brainstorming to solve a stated mathematics question with your students.
2. Explain the implications of engaging learners in a role play.

UNIT 3: TEACHING METHODS 2

The following are some general methods often employed in teaching mathematics: Cooperative learning, Exposition, Guided discovery, Games, Laboratory approach, Simulations, Problem solving and Investigations. For effective teaching use a combination of any of these methods.

Learning outcome(s)
By the end of the unit, the participant will be able to:
- explain the merits and demerits of cooperative learning;
- explain expository method of teaching;
- explain the implications for using mathematical games;
- explain discovery and inquiry teaching methods;
- distinguish between problem solving and investigation;
- explain brainstorming, role play and heuristic teaching methods.

SESSION 1: COOPERATIVE GROUPS

This session deals with cooperative learning activities. Cooperative learning definitions vary. According to Linblad (1994), cooperative learning in its purest form is merely where a few people get together to study something and produce a single product. But because self-reliance is every bit as important a skill to master as cooperative relationships good teachers will continue to emphasize the importance of individual effort and accountability at the same time that they use cooperative learning techniques.

Learning outcome(s)
By the end of the session, the participant will be able to explain what is meant by cooperative learning and the advantages of using cooperative learning groups in mathematics classrooms.

Definition of Cooperative Learning
Cooperative learning involves two or more students working toward a common goal. Distinct goals are usually set for the group to reach. All students in the group are to help to attain their group goals. Because students differ in their abilities, differentiated assignments may be required to allow all of them to contribute to attaining the group goal. The cooperative learning programme requires all group members to succeed. Performance is based on total group success because it prompts all group members to help their fellow. Group success depends on all of its members' success.

Through cooperative learning, teachers can allow students the opportunity to benefit from the strengths of their peers. Many teachers integrate cooperative learning into their instruction by allowing students to complete assignments in pairs or small groups. For optimal effectiveness, teachers should carefully select groups, pairing more capable students with less capable ones. Although for group harmony, students may be allowed to select their own group members, teachers should ensure that there is diverse range of capabilities in each group, as much as possible.

The method is more a method of organization than a specific teaching strategy.

i. Students work in small groups (4-6) – encouraged to discuss and solve problems
ii. Students are accountable for management of time and resources both as individuals and as a group.
iii. Teacher moves from group to group giving assistance and encouragement, asking thought-provoking questions as the need arises.
iv. Group work is usually reported to the entire class and further discussion ensues.

v. It allows students to work together as a team fostering co-operation rather than unhealthy competition.
vi. It provides for students – students discussion, and social interaction.

Johnson and Johnson (1989) report that cooperative learning has five basic elements: (i) positive interdependence; (ii) face to face interaction; (iii) individual accountability; (iv) collaborative skills and (v) group processing. Positive interdependence means that each student's success (grade) depends on a group's grade. The converse of this principle is also true; the group's success depends on each member's mastering of the material being tested leading to individual accountability. There is interaction among the group members where they share ideas, argue with each other till some consensus is reached. They pull resources together for the common goals. Group processing is the group's discussion and assessment of their progress.

Advantages of Cooperative Learning

Cooperative learning teams can increase student self-esteem if clear norms are established and the sharing and respecting of each member's ideas are valued. Although cooperative learning programmes seem especially suited for low achievers, studies show that high, average and low achievers gain equally from cooperative learning experiences (Manning & Lucking, 1991). Cooperative learning leads to a more pro-social orientation among students. Cooperative work promotes higher achievement, develops social skills in learners and puts the responsibility for learning on the learner.

i. Cooperative learning makes it possible for students to benefit from their classmates' knowledge and thoughts.

ii. Cooperative learning removes competition between and among students.

   The assessment used with cooperative learning motivates students yet protects less capable students from challenges beyond them. Cooperative learning teaches students to cooperate with others. Since the team's score is a sum of the team members' scores, each participant is encouraged to help fellow team members.

iii. Cooperative learning is motivational. It encourages all students to do their best.

   Co-operative learning involves discussion and exchange of ideas. If activities which embody mathematical concepts are done in pairs or small groups, it is expected that students will talk about what they are doing. In such situations they will be talking about mathematics, as embodied in these materials and activities. This has a number of benefits:

   a. Students are putting their thoughts into words, in a mathematical situation. This is an important first step towards putting mathematics on paper, which is more difficult.

   b. Mathematical activities and games in which success depends largely on mathematical thinking enhance cooperative learning.

iv. The activities based on physical embodiments of mathematical ideas provide shared sensory experiences which ensure that there is shared resource for students' discussion. Other activities, in which symbols are used, are more abstract. In this case the shared experience is partly at a symbolic level, but more importantly at a mental level in the form of shared mathematical ideas and experiences.

The **implications** of cooperative learning for teaching **include**:

i. It promotes cooperation among students. Team work is encouraged and students gain from each other.
ii. Students learn to accept responsibility for their own learning (autonomy).
iii. It reinforces understanding –each student can explain to other group members.
iv. It calls for a change in teachers' role from leader to facilitator and initiator.

The l**imitations** are:

i. The method requires more careful organization and management skills from the teacher.
ii. It demands careful pre-planning and investment of time and resources in preparing materials.
iii. Requires that teachers are skilled in identifying and selecting mathematically rich moments as they occur during group discussions to share with the whole class.

Reflection

1. What is cooperative learning?
2. State and explain three advantages of cooperative learning
3. State and explain two implications of cooperative learning to the teaching of mathematics.

SESSION 2: EXPOSITORY METHOD

Expository teaching is basically a **direct** instruction. It is a lecture, presentation or telling strategy used during instruction. The teacher presents students with the subject matter rules and provides examples that illustrate the rules. The examples can be prerequisite information, application of the rules, context through historical information, and pictorial relationships. The examples are to give contextual elaboration and to help students see the subject matter from many different perspectives. This session discusses what expository teaching method is, its advantages, disadvantages and implications for teaching.

Learning outcome(s)

By the end of the session, the participant will be able to state and explain what expository method of teaching is, its merits, demerits and limitations.

Meaning of Expository Teaching

Expository teaching usually involves presenting clear and concise information in a purposeful way that allows students to easily make connections from one concept to the next. The structure helps students to stay focused on the topic at hand. The teacher is in control of presenting the subject matter and directs the students through the lesson, focusing students' attention on the key points of the subject.

Often times, when students are discovering information on their own, they can get distracted and confused by unnecessary information and have difficulty determining what's important. This is why expository instruction is one of the most common instructional strategies. Most educators believe students learn new concepts and ideas better if all of the information they need to know is laid out before them.

The teacher often gives both the principles and the problem solutions, and presents the students with the entire content of what is to be learned. Usually, teacher starts with a definition of concepts or principles, illustrates them, and unfolds their implications. It is sometimes called deductive teaching. Expository teaching has been identified with rote learning. The students, presumably, can only memorize the lectures by constant review and repetition.

Expository teaching, however, can present a rich body of highly related facts, concepts, and principles which the students can learn. There is an introduction and overview (the advanced or conceptual organizer, at times at a higher level of abstraction, generality, and inclusiveness than the content of the lesson). This is followed by more specific information and detail in a hierarchically arranged sequence. Moving from the general to the specific allows students to understand the increasingly detailed explanations of the information and link those explanations to information that was presented previously as part of the general overview. Programmed instruction is a form of expository teaching (Glaser, 1966). Expository teaching involves a clear and proper sequenced explanation by the teacher of the idea or concept.

Expository teaching builds on the ideas of:

i. Robert Gagné who proposed that teaching begins at the lowest level which serves as a prerequisite for a higher level (building Map of prerequisites and Learning hierarchies).
ii. Jerome Bruner, who advocated that mathematics, should be represented in at least 3 ways – enactive, iconic, and symbolic.
iii. Zoltan P. Dienes, who focused on dynamic principle which suggests that play should be incorporated in the teaching of mathematics concepts.

Merits of Expository Teaching

i. Through proper expository teaching students can proceed directly to a level of abstract understanding that is qualitatively superior to the intuitive level in terms of generality, clarity, precision and explicitness.

ii. Teachers present information to their students in a purposeful way that allows students to (easily) make connections from one concept to the next. Students receive the information from an expert. The use of an advanced organizer at the introductory stage helps students stay focused on the topic at hand.

iii. Expository teaching is more efficient and takes less time than discovery learning. It offers the student the best opportunity to obtain an organized view of the discipline he is studying because the teacher can organize the field much more effectively for learning than the novice student can.

iv. This method is beneficial for the auditory learner who does best when listening to instruction.

Limitations

This method's drawbacks are that (i) the teacher is in complete control of every aspect of the lesson and the students are not able to take an active part in their education. (ii) Poor expository teaching leads to passive learners; (iii) retention and utilization of learning may be curtailed.

(iv) It does not adequately cater for individual differences, and (v) it can be teacher-dominated rather than child-centered.

Implications for Teaching

Expository teaching is a fast and efficient way of giving information; relatively easy to organize and often requires little teacher preparation; possible for teacher to motivate with enthusiastic and lively discussion. The lesson can be regulated according to the student's response.

Reflection

1. What do you understand by the expository method of teaching?
2. State and explain three merits and three demerits of the expository method of teaching
3. Explain the implication of the expository method in the teaching of mathematics

SESSION 3: MATHEMATICS GAMES

Games, sometimes called **Recreational mathematics**, are mathematical problems that are fun. They can motivate students to learn mathematics and can increase enjoyment of mathematics. This session introduces you to what mathematical games are and the importance of using games to teach mathematics.

Learning outcome(s)

By the end of the session, the participant will be able to justify the use of mathematical games in teaching mathematics.

Meaning of Mathematical Games

A **mathematical game** is an activity with rules performed either alone or with others, for the purpose of entertainment. In many games, the objective is to win by defeating the other player or players or being the first to reach a specified goal, while in others, role-playing or cooperation is emphasized. Games are predominantly used to practice and reinforce basic skills; additionally it can be used to introduce new concepts and develop logical thinking.

A **mathematical game** is a multiplayer **game** whose rules, strategies, and outcomes are defined by clear mathematical parameters. Often, such games have simple rules and match procedures, such as Tic-tac-toe, Number game and Three-in-a-line. Generally, mathematical games need not be conceptually intricate to involve deeper computational underpinnings.

Gough (1999) states that "A 'game' needs to have two or more players, who take turns, each competing to achieve a 'winning' situation of some kind, each able to exercise some choice about how to move at any time through the playing". The key idea in this statement is that of 'choice'. In this sense, something like Snakes and Ladders is NOT a game because winning relies totally on chance. The players make no decisions, nor do they have to think further than counting. There is also no interaction between players - nothing that one player does affect other players' turns in any way.

Oldfield (1991) says that mathematical games are 'activities' which:

- involve a challenge, usually against one or more opponents;
- are governed by a set of rules and have a clear underlying structure;
- normally have a distinct finishing point;
- have specific mathematical cognitive objectives.

Games can motivate students to improve skills that are usually learned by rote. In 'Number Bingo,' players roll three dice, then perform basic mathematical operations on those numbers to get a new number, which they cover on the board trying to cover 4 squares in a row.

Mathematical games differ from mathematical puzzles in that mathematical puzzles require specific mathematical expertise to complete, whereas mathematical games do not require a deep knowledge of mathematics to play. The arithmetic core of mathematical games is not readily apparent to players untrained to note the statistical or mathematical aspects. Some mathematical games are of deep interest in the field of recreational mathematics. To analyze a game numerically, it is particularly useful to study the rules of the game insofar as they can yield equations or relevant formulas. This is frequently done to determine winning strategies or to distinguish if the game has a solution.

A **puzzle** is something, such as a game, toy, or problem that requires ingenuity and often persistence in solving or assembling. It is something that baffles or confuses. It is an enigma.

Benefits of Using Games

Davies (1995) identified the following advantages of using games in a mathematical programme.

i. Meaningful situations - for the application of mathematical skills are created by games

ii. Motivation - students freely choose to participate and enjoy playing

iii. Positive attitude - Games provide opportunities for building self-concept and developing positive attitudes towards mathematics, through reducing the fear of failure and error;

iv. Increased learning - in comparison to more formal activities, greater learning can occur through games due to the increased interaction between learners, opportunities to test intuitive ideas and problem-solving strategies

v. Different levels - Games can allow children to operate at different levels of thinking and to learn from each other. In a group of learners playing a game, one child might be encountering a concept for the first time, another may be developing his/her understanding of the concept, a third consolidating previously learned concepts

vi. Assessment – learner's thinking often becomes apparent through the actions and decisions they make during a game, so the teacher has the opportunity to carry out diagnosis and assessment of learning in a non-threatening situation

vii. Home and school - Games provide 'hands-on' interactive tasks for both school and home

viii. Independence - Learners can work independently of the teacher. The rules of the game and the learners' motivation usually keep them on task.

ix. Few *language barriers* - an additional benefit becomes evident when learners from different language-speaking backgrounds are involved. The basic structures of some games are common to many cultures, and the procedures of simple games can be quickly learned through observation. Learners who are reluctant to participate in other mathematical activities because of language barriers will often join in a game, and so gain access to the mathematical learning as well as engage in structured social interaction.

Implications for Teaching

Games are usually highly motivating and are more likely to generate greater understanding and retention. Games are an active approach to learning and students often enjoy playing games.

Games foster good attitudes to mathematics. Mathematical games in which success depends largely on mathematical thinking enhance cooperative learning. The rules for the games are largely mathematical, so whether a move is allowable or not depends on agreement about what is correct or incorrect mathematically. In this way, students correct each other's mistakes in a way which is much less threatening than being told one is wrong by a teacher. Trying to justify, or disagree with, a move on mathematical grounds means explaining oneself clearly, and this requires one to get these ideas clear in one's own mind.

The method or methods used in any particular context are largely determined by the objectives that the relevant educational system is trying to achieve. Methods of teaching mathematics include the following:

Alridge & Badham (1993) give the following tips for successful classroom games:

i. Make sure the game matches the mathematical objective
ii. Use games for specific purposes, not just time-fillers
iii. Keep the number of players from two to four, so that turns come around quickly
iv. The game should have enough of an element of chance so that it allows weaker students to feel that they a chance of winning
v. Keep the game completion time short
vi. Use five or six 'basic' game structures so the children become familiar with the rules - vary the mathematics rather than the rules
vii. Send an established game home with a child for homework
viii. Invite children to create their own board games or variations of known games.

Limitations

Collection and construction of materials for game is time consuming. Students might be so engaged in the game and not realize the mathematical ideas, principles or concepts inherent in the game being played. Also, the approach is not suitable to all areas of the syllabus and can be quite ineffective in very large classes.

Reflection

1. What is a mathematical game?
2. State and explain three advantages of mathematical games.
3. Explain the implications of using mathematical games for the teaching of mathematics.

SESSION 4: DISCOVERY METHOD AND INQUIRY METHOD OF TEACHING

Constructivist teaching requires that teachers engage students in experimentation and allow them to use the results of these experiments to reach their own conclusions. The experimentation leads students to make their own discoveries in mathematics. This method can prove effective in teaching many mathematics concepts. In this session, we will learn about discovery and inquiry approaches to teaching mathematics.

Learning outcome(s)

By the end of the session, the participant will be able to explain discovery and inquiry methods of teaching mathematics.

Discovery Teaching

There are two forms of discovery method – guided discovery and free discovery.

Guided Discovery is an approach in which the teacher presents a series of structured situations to students. The students then study these situations in order to discover some concept or generalization. As opposed to exposition, the learner is not told the rule or generalization by the teacher and then asked to practice similar problems. Instead students are asked to identify the rule or generalization. Not all students find it easy to discover under all circumstances and this may lead to frustration and lack of interest in the activity. To avoid this, it may be necessary to have additional clues to assist the students, through guidance, to discover the rule or generalization.

The guided discovery method of teaching is good because students are provided with structured guidelines for assignments but allowed to discover details on their own. Students are given a set of guidelines to follow and then collect information and data on their own. But it is especially limiting for students who think outside the box and need to discover things on their own.

Free Discovery

Free Discovery is excellent because students are able to be a part of their own education. They are free to use their critical thinking skills to discover the world around them. However the drawback with this method is that it is not structured and most students need structure and guidelines to follow so that they can stay on task.

In discovery learning the concern to teach the techniques of discovery overrides the concern for learning the unifying principles of a discipline. The best methodology is to mix teaching methods. The teacher starts the lesson off with visuals and a short lecture, and then the class discusses the material by asking and answering questions. The students are given guidelines to follow such as prompts that are chosen ahead of time or sometimes they come up with their own prompt. After everything is in place students are given the freedom to discover the world around them.

Examples

1. Discovering the value of pie to be 3.14 by measuring the distance round cylindrical objects and their diameters and finding the ratio of the circumference to the measure of the diameter.
2. Discovering sum of angles in a triangle by measuring the angles of a number of different sized triangles and then finding the sum of the three angles in each triangle.
3. Discovering area of a circle by splitting a circular cut outs into small sectors and rearranging the sector to form an approximate rectangle and then finding the area of the resulting rectangle/parallelogram.
4. Discovering sum of angles in a polygon by drawing non intersecting diagonals in a number of polygons and finding out the number of triangles in each polygon.

Limitations

Guided discovery is time consuming for teacher to organize – some students may never discover the concepts or principles. It demands a fair amount of expertise from the teacher. It requires technical expertise (i.e. how best to organize or present the subject) and a good knowledge of the students (i.e. how much help/guidance should be given).

Inquiry methods in which specific problems are set for investigation, bearing in mind the resources available, but freedom in the method of solving the problem is allowed.

Inquiry methods though effective can be time-consuming requiring decisions from the teacher on the best mix of methods to use to achieve his instructional objectives. The use of these methods requires very careful planning in advance by the teacher (Perrot et al 1977). **Inquiry Approach** requires the learner to seek knowledge through his/her own effort by carrying out investigations, drawing generalizations/making discoveries and finding solutions.

Activity:

1. Identify the problem
2. Formulate a hypothesis (guess)
3. Collect data and test the hypothesis
4. Draw conclusion(s)
5. Apply conclusion to make generalization
6. Revise hypothesis

Advantages of Inquiry method

1. Critical and problem-solving skills are enhanced
2. It necessitates in-depth understanding and involvement of learners in solving problems
3. Learner gains a great deal of independence to plan and decide how to carry out investigations. Teacher plays facilitator role (guidance)
4. Learners learn from experience (success and failure)
5. It stresses students' ability to observe, collect information, analyze information, anticipate expected outcomes and evaluate conclusions and decisions.

Limitations

1. Enough exposure is needed by the user.
2. Problem may require materials, time and equipment beyond available resources.

Reflection

1. Explain what is meant by guided discovery method of teaching mathematics.
2. Distinguish between "guided discovery" and "free-discovery" as methods of teaching mathematics.
3. State and explain three limitations of the guided discovery method.
4. State and explain two advantages and two disadvantages of inquiry method of teaching topic.

SESSION 5: PROBLEM SOLVING AND INVESTIGATION

Mathematical tasks can be classified as Exercise, problem solving and investigations. Exercise deals the reinforcement of mathematical skills by completing large numbers of exercises of a similar type, such as adding vulgar fractions or solving quadratic equations.The ability to investigate and solve problems is at the heart of mathematics. Mathematics is only 'useful' to the extent that it can be applied to particular situation and it is the ability to apply mathematics to a variety of situations to which we give the name 'problem solving'. Mathematical investigations and problem solving are often used interchangeably. Problem solving and investigation form the process part of mathematics that is crucial but has often been overlooked in the past in favour of computation skills. Problem solving and investigation involve the cultivation of mathematical ingenuity, creativity and heuristic thinking by setting students open-ended, unusual, and sometimes unsolved problems. The problems can range from simple word problems to problems from international mathematics competitions Problem solving is used as a means to build new mathematical knowledge, typically by building on students' prior understandings. While problem solving has a definite goal, investigation may lead to other problems leading finally to some other results. This session discusses problem solving and investigation as methods of teaching mathematics.

Learning outcome(s)

By the end of the session, the participant will be able to distinguish between problem solving and investigation as methods of teaching mathematics.

Problem Solving

The word problem comes from the Greek word ''problema'' meaning ''something thrown forward''. A problem is a question raised for inquiry, consideration or solution. For a question or an exercise to be a problem, the question or the exercise must present a challenge that cannot be resolved by some routine procedure known to the student. Note that while a question may be a problem to one student, the same question may not be a problem to another student. A problem is therefore relative to the problem solver. Problem solving is therefore the process of the problem solver accepting the challenge offered by a problem and taking a decision to solve the problem that confronts him using a non-routine method.

In using problem solving method, a mathematics task is presented first to the class in a natural way. The task needs to be worthwhile and stimulating to the students. The students are set to work on the task as individuals or groups. The teacher serves as a facilitator. The results are discussed as a class or with the individual students or groups and feedback is given by teacher.

A key focus in teaching through problem solving is the process by which a student obtained a particular answer. As such, students are encouraged to share their thinking with the class and provide justification for *what* they did, *how* they did it, and *why* their approach is valid.

Problem solving is a suitable approach in the teaching of mathematics. It develops in the learners the ability to recognize, analyze, solve and reflect upon the problematic difficulties.

In the problem-solving method of teaching, the problem itself is the crux of the problem. A problem is a sort of obstruction or difficulty which has to be overcome to reach a goal. The problem method aims at presenting the knowledge to be learnt in the form of a problem. The problems are set to the students in a natural way and it is ensured that the students are genuinely interested to solve them. The solutions to the problem always come from the students; the teacher remains in the background and directs or guides the students' activity from that position.

The procedure involves:

1. Recognizing the problem or sensing the problem.
2. Interpreting, defining and delimiting the problem.
3. Gathering data in a systematic manner.
4. Organizing and evaluating the data.
5. Formulating tentative solutions
6. Arriving at the correct or true solution
7. Verifying the result.

Merits of problem-solving method

i. Students are actively involved in the learning process.
ii. It is based on actual observations, thinking and experimentation.
iii. It enhances understanding.

Demerits of problem-solving method

i. It is time consuming method.
ii. It is an intellectual approach in learning.
iii. Teachers' burden becomes heavier, as a lot of preparation is needed.
iv. The method does not suit the students of lower classes as this is a scientific method.

Some Problem-Solving Challenges

1. **Frog in the Pond**
   There is a circular pond with a circumference of 600 meters. Dead in the centre of the pond is a frog on a stepping stone. If the average leap of a frog is two and a quarter metres and there are plenty of other stepping stones on which to jump, what is the minimum number of leaps it will take for the frog to jump completely out of the pond?

   [Read the question carefully and notice that the frog is dead and so cannot move. This funny challenge gets students to do the mathematics with an "out of the box" answer. You can change the numbers and units for different learners, but the answer will always be the same. Students will do the math and love the answer.]

2. **Cute Little Bookworm**
   There is a three-volume set of books sitting on a bookshelf. The front and back covers of the books are each one-eighth of a centimetre thick. The page section inside each book is exactly two centimetres thick. A cute little bookworm starts eating at page one of Volume One and eats, in a straight line, through to the last page of Volume Three. How far will the little worm travel?

   [Be careful! Try using three books. Look at the spines of the books. Books in Volumes are arranged in a special way in libraries. The answer is not 6.5 cm. Volume One is the leftmost book. Therefore, page one of Volume One will be on the right side (or inside portion of the books when looking at the spines) of the book, not left and the last page of Volume Three will be on the left side of volume three. Therefore, the worm only eats through one cover of Volume One ($\frac{1}{8}$), two covers and the page section of Volume Two ($\frac{1}{8}+\frac{1}{8}+2$), and one cover of Volume Three for a total of ($\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+2+\frac{1}{8}=2\frac{1}{2}$). Answer is 2.5 cm.]

3. **Handshake Problem**

An interesting problem solving situation deals with a group of people in a room shaking hands. If persons A and B are the only ones in a room, and if they shake hands, there is only one handshake and it counts for both of them. If there are 10 people in the room, how many handshakes would be generated?

Let us start from a simple case of two people in the room, there would be one handshake. Three people, (A, B, and C) in the room generate 3 handshakes (AB, BC, AC). There would be no need to consider handshakes for BA, CB, CA, because each handshake counts for both people.

Now can you complete the table below?

Number of people in a room	**Number of handshakes**
2	1
3	3
4	
5	
6	
7	
8	
…	
…	

The figure below shows how the handshake problem could be represented geometrically. Each vertex stands for a person and each line segment connecting vertices represents one handshake between the two people.

Now show figures for 5, 6 and 7 people shaking hands and show the total number of handshakes under the given rules.

A formula can be developed to show the number of handshakes needed for any number of people given the stated rules. Some clues for finding the formula can be found by studying Table 2. Two people result in one handshake. That could mean divide the number of people in half to get the number of shakes.

Table 2

People	Shakes
2	1
3	3
4	6
5	10
6	15
…	…

Test the idea with three people. The first conjecture fails. We could say that we just take the number of people as the number of handshakes, but we know that will not work with two people.

We need to take a look at more numbers and seek a pattern. The pattern in Table 2 doesn't seem obvious. What if we consider each handshake as counting for each individual? That is, A and B would generate two handshakes, one for A, and another one for B. Similarly, with A, B, and C, we would have six handshakes: AB, BA, AC, CA, BC, and CB.

Build a new table as shown in Table 3 by adding a column for this new rule:

Table 3

People	Shakes	Shakes for each
2	1	2
3	3	6
4	6	12
5	10	20
6	15	30

There is a clue. Counting each handshake twice doubles the numbers in the first and third columns. The third column entry is the first column entry multiplied by one less than the first column entry. That is, 2=21,6=32,12=4(3) $2 = 2(1); \quad 6 = 3(2); \quad 12 = 4(3)$ and so on.

Now compare the third column entries with their respective second column entries. In each case the third column entry is twice the second column entry.

Now create a formula that shows how to determine the number of handshakes given any number of people.

You might notice that these are the triangular numbers that are generated by the rule $T_n = \frac{n(n-1)}{2}$ as the *n*th triangular number.

We can transform this to be $H_n = \frac{n(n-1)}{2}$ as the number of handshakes involving *n* people.

Thus for 10 people, there are $H_{10} = \frac{10(10-1)}{2} = \frac{10 \times 9}{2} = 45$ handshakes.

4. Pyramid of Numbers

1
11
21
1211
111221
312211
What are the next 2 lines?

Investigations

The idea of an investigation is fundamental both to the study of mathematics and also to the understanding of the ways in which mathematics can be used to extend knowledge and to solve problems. An investigation is a form of discovery. At its best, students will define their own problems, set procedures and try to solve them. In the end, it is crucial for the students to discuss not only the outcomes of the investigation but also the process pursued in trying to pin down the problem and find answers to the problem. As opposed to the guided discovery lesson where the objectives are clear, an investigation often covers a broad area of mathematics objectives and includes activities which may have more than one correct answer. In investigation, the process of solving a task is as valued as the product (answer).

The method is suitable for mixed ability groups and promotes creativeness. It can be intrinsically satisfying to students.

Students generally follow the following steps in investigation:

1. Initial problem

2. Data collection

3. Tabulate or organize the data

4. Making and testing conjectures

5. Try new concept if first conjectures are wrong

6. Attempt to prove a rule

7. Generalization of the rule

8. Suggest new or related problems

More able students can develop their creativity doing investigations and can perform all of the eight features. Less capable students may only be able to carry out the first 3 stages.

Limitations

Investigations require a high degree of teacher input. Investigations can be difficult to fit into the conventional mathematics syllabus and they can be time consuming.

Example: Why is it so?

- Write down any three numbers less than ten, e.g. **3, 4** and **7**.
- Make all the six possible 2-digit numbers using these numbers: **34, 37, 43, 47, 73, 74**.
- Find their sum. (308)
- Calculate the sum of the original numbers (14).
- Divide the first total by the second. $14\overline{)308} = 22$
- Answer? 22

Repeat the operations for **other** combinations of numbers.

Algebraically: Let a, b, c represent the numbers less than ten.

$\therefore\ 10a + b$	
$10a + c$	
$10b + a$	$22a + 22b + 22c$
$10b + c$	$= 22(a + b + c)$
$10c + a$	$\therefore\ \dfrac{22(a + b + c)}{(a + b + c)}$
$\underline{10c + b}$	
$\underline{22a + 22b + 22c}$	$= 22$

Reflection

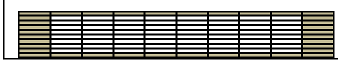
1. Distinguish between problem solving and investigation as methods of teaching mathematics.
2. State and explain three merits and three demerits of problem solving as a technique of teaching mathematics.
3. Explain the implications of using problem solving in teaching mathematics.
4. **The Border Problem:** Look at the grid shown.

a. Without counting 1-by-1 and without writing anything down, calculate the number of shaded squares in the 10 by 10 grid shown. You can invite students to share the various strategies they used.

b. What will be the number of shaded squares in:

i. 8 by 8; (ii) 15 by 15; (iii) 100 by 100 square grid?

c. Determine a general rule for finding the number of shaded squares in any similar *n*-by-*n* grid.



SESSION 6: HEURISTICS METHOD

Other teaching approaches include Laboratory Approach and Independent Studies. Laboratory approach is defined as "learning by doing". More often than not it involves students playing and manipulating concrete objects in structured situations. The purpose is to build readiness for the development of more abstract concepts – often combined with guided discovery methods. Its **implications** for teaching is that it has the support of theorists [behaviourists – Pavlov, Skinner; Developmentalists – Bruner,…]. In an organized situation, students are able to proceed at their own rate. The students develop their own spirit of inquiry. The approach is especially useful for younger children and small class sizes.

Its **limitations** are as follows:

i. It requires a good supply of materials and suitably designed classrooms.
ii. It also demands a fair amount of teacher preparation and creativeness.

Independent Studies refer to a common situation where each student carries out a given activity independently e.g. solving of a mathematics problem. Another situation involves a completely open-ended choice of individual activity. Here objectives are hidden from the students until the tasks are complete. This session deals with one other teaching method called Heuristics.

Learning outcome(s)

By the end of the session, the participant will be able to state and explain heuristics method of teaching mathematics.

Heuristic Method

The term heuristic is derived from a Greek word, which means "I find". The heuristic method is a technique by which pupils learn to reason for themselves. In this method, the child is put in the place of a discoverer. It aims at removing the shortcomings attributed to lecture method. Staunch supporters of this method are of the opinion that every child should be made a discoverer and inventor. The teacher allows the child to help himself with reasoning and argument.

Procedure:

i. The teacher presents practical problems to the pupils.
ii. The teachers role is not to solve the problem for the pupils but to enable them solve the problems for themselves.
iii. The teacher is required to encourage, help or guide the pupil by asking questions.
iv. The extreme form of this method is that the teacher stands aside as an onlooker and the child selects his own path and proceed.
v. The child needs guidance.

In fact, the success of this method depends largely on good questioning. The teacher no longer teaches; he guides. Also, the learner no longer listens; he finds.

False Heuristics	True Heuristics
1. Is it true that a square has all sides equal?	What do you know about the sides of a square?
2. Do you remember that the area of a rectangle = length x breadth?	How do you calculate the area of a rectangle?
3. Tell me whether the profit or loss % is calculated on the cost or selling price?	On which price do we calculate profit or loss %?

Merits

a. The student becomes an active participant in the learning process
b. The student thinks for himself or herself and does not merely listen for information.
c. The student acquires a real understanding and clear notion of the subject. It gives him or her a complete mastery of what he or she has learnt.
d. It creates in them a spirit of enquiry.
e. The student becomes self-reliant
f. Memorization work becomes light.
g. It gives students happiness and mental satisfaction and encourages them towards further achievement.

Demerits

a. It demands extraordinary labour and special preparation from the teacher, who is already overburdened.
b. It is a slow method.
c. Every teacher may not be able to use it successfully. The teacher must be gifted with the heuristic spirit.
d. Every child cannot be expected to be a gifted discoverer. The immature child may have difficulties.

Reflection

1. Explain how you will employ the heuristics method to teach a mathematics topic identified by you.

UNIT 4: BASIC TEACHING FUNCTIONS

Decisions that teachers often make with regard to students' learning and appropriate teaching strategies to employ are concerned with three basic teaching functions - *planning*, *implementation* and *evaluation*. Control over teaching and learning can be exercised most effectively in three ways:

i. Substantively, by showing concern for the structure of the body of subject-matter
ii. Pragmatically, by employing suitable principles of ordering the sequence of subject-matter and constructing its internal logic and organization.
iii. Arranging appropriate practice trials (Ausubel, 1965; Bruner, 1960; Gagne`, 1965).

In this unit we will discuss the three basic teaching functions, scheme of work, planning for mathematics instruction, objective statements for specific lessons, guidelines for lesson planning, and format for writing lesson plan at SHS level.

Learning outcome(s)
By the end of the unit, the participant will be able to**;**

- explain the terms planning, implementation and evaluation and the supporting skills for each function;
- prepare a scheme of work;
- explain the importance of planning for a mathematics instruction;
- write SMART objective statements for mathematics lessons;
- identify specific guides for preparing lesson notes;
- use the format to write a lesson plan for specified SHS topics.

SESSION 1: PLANNING, IMPLEMENTATION AND EVALUATION

Welcome to this session. We will discuss the three basic teaching functions that teachers usually make decisions about in order to ensure effective teaching and learning. These are planning, implementation and evaluation.

Learning outcome(s)
By the end of the session, the participant will be able to identify and explain the three basic teaching functions.

Planning
Planning is a vital element in teaching, since the whole decision making model is based on this skill. Cognitive learning theorists indicated that the amount and rate of learning is influenced by the nature of the subject matter itself, the way it is broken down and the order in which it is presented (Anderson and Ausubel, 1965).

This function requires teachers to make decisions about the learners' needs, the most appropriate goals and objectives to help meet those needs, the motivation necessary to attain their goals and the most appropriate teaching strategies to use (i.e. what activities to use, how to sequence these activities, and the language to use in conveying the mathematical ideas and concepts to students). The teacher considers students' progress, availability of resources, equipment and materials; the time requirements of activities etc. (Perrott, et al. 1977). Teaching skills that support the planning function are diagnosing students' needs, setting goals/objectives and determining appropriate learning activities.

Teaching is not simply standing in front of a group of students and telling them how to do things or simply checking students' answers on tests or assignments. Teaching is selling. The teacher of mathematics must be an advocate of the field. He must have belief, excitement and enthusiasm about what is being covered. Good classes do not just happen; they are carefully planned and orchestrated. There can be deviations from the plan but the framework is laid out well ahead of time. There are three axioms to consider:

a. Know the content being presented.
b. Know more than the content being presented.
c. Teach from the overflow of knowledge using knowledge about how students learn.

These imply good teaching does not happen by accident. Rather, they are carefully orchestrated through effective planning and prior organization. It should be done well in advance to allow time for ideas to germinate and blend in your subconscious. It also provides an opportunity to connect topics from different lessons throughout the course. Even though all classes are given the same objective to complete within a given time frame, variation of presentation styles, relating the subject matter to background material, calling on students' strengths established earlier in the curriculum, and use of technology can all provide extra time that permits flexibility for teaching. All these are accomplished when good planning is done.

Implementation

Implementation involves teacher carrying out his/her plans especially teaching methods, strategies and learning activities. This occurs when teacher interacts with the students. Teaching skills that support this function are: presenting, explaining, listening, introducing, demonstrating, eliciting responses and achieving closure.

Evaluation

Evaluation requires decisions about suitability of objectives and teaching strategies linked to them, and eventually whether or not the students are achieving what the teacher intended. Teaching skills which support this function are specifying the learning objectives, describing the information needed to make such an evaluation, recording, analyzing and forming judgments. Examine the results of your teaching and decide how well you handled each teaching function. This feedback makes you decide to make new plans or try different implementation strategies. It is assumed that:

1. Teaching is goal directed i.e. some change in students thinking or behaviour is sought.
2. Teachers are active shapers of their own behaviour- they make plans, implement them and continually adjust to new information concerning the effects of their actions.
3. Teaching is a rational process which can be improved by examining its components in an analytic manner.
4. Teaching behaviour can affect students' behaviour and learning.

You can think about evaluation as occurring at three key stages during the lesson planning, lesson implementation, and post-implementation. In the selection of lesson objectives and methods for any topic, you should evaluate the choices being made as to the suitability or otherwise of selected objectives and methods. During the lesson implementation you should engage in ongoing evaluation as to whether you are likely to achieve stated objectives, whether there is a need to vary or change your instructional strategy based upon how students are responding to questions. After the lesson delivery, there is a need to still determine among others, what objectives were fully, partially or not achieved. What could have accounted for students' successful learning of stated learning behaviours? Is there a need to re-order your instructional objectives or even consider a particular topic before introducing what you just taught?

You should have realized by now that teaching is a complex activity requiring constant evaluation. In order to be an effective teacher you need to be a reflective practitioner where you are constantly thinking through your actions, asking a series of question as to the soundness and effectives of your choices, and seeking to constantly improve on your teaching and your students' learning experiences. In other words, an effective teacher should be constantly engage in a process of metacognition – possess the ability to think about your own thinking (Hammerness et al., 2005). Flavell (1979) identified two aspects of metacognition as follows: 1) *metacognitive knowledge* – "understanding one's own thinking and developing strategies for planning, analysing and gaining more knowledge" and 2) *metacognitive regulation* – "being able to define learning goals and monitoring one's progress in achieving them" (cited in Hammerness et al., 2005, p.376). In order for you to develop such an expertise (adaptive expert – who can make changes to instruction irrespective of the complexity of teaching through reflection) there is a need for the possession of three types of knowledge according to Cochran-Smith and Lytle (1999): 1) knowledge *for* practice; 2) knowledge *in* practice; and 3) knowledge *of* practice. Knowledge *for* practice refers to the knowledge you are gaining at the moment as you prepare to become SHS mathematics teachers. All the various courses you are taking either in mathematics (content), pedagogy (methods of teaching), theories of students' learning (psychological basis of teaching mathematics) are geared towards preparing you for the classroom. Experts in mathematics and mathematics education, through research, have identified a body of knowledge that they believe you would need in your (future) classrooms.

Knowledge *in* practice or "knowledge in action" (Hammerness et al., 2005, p. 382) refers to knowledge that effective (successful) teachers have learned or acquired through their practice. That is, as you teach you are expected to gain some insights into teaching based upon the students you have interacted with over the years, classroom environment, school-level climate, etc. This knowledge is therefore knowledge about your own experiences. As such, knowledge in practice is based upon a teacher's unique classroom or school situations acquired through reflection on his or her experience. What we expect from you is that while you make use of the knowledge being acquired through your degree programme (knowledge for practice) you should also engage in continuous reflection of your teaching and students' learning so that you can add to what you have learned, learning now or about to learn. You can consider whether it is appropriate to teach topic A before topic B as suggested in the core mathematics syllabus depending upon your students' characteristics, for example,

The third type of knowledge is about the "relationship between knowledge and practice and the theoretical aspects of both…It emphasizes the role of the teacher in constructing knowledge and learning and growing through that process" (Hammerness, 2005, p.383). In effect, you should learn from your classroom situations and from your colleagues. Your learning does not end once you graduate. You are expected to continue to learn from your classroom and school contexts and devising ways that are responsive to your unique situations such that students' learning improves.

Korthagen and Vasalos (2005) identified nine questions to consider when reflecting on practice (instruction) which is modified in the table:

1. What was the instructional objective?	
1. What did you intend for students to learn (concepts, procedures, skills) and be able to do (capability)?	5. What did the students want to learn?
2. What did you do? What strategies did you employ?	6. What did the students actually do?
3. What made you select an objective and particular method(s)	7. What were the students thinking?
4. How did you feel after the lesson? Successful? Not successful? Elated?	8. How did the students feel or react towards the lesson?

To acquire teaching skills prospective teachers need to:

1. Identify the skill, study and observe the skill, know the purpose of teaching it and how it will benefit your students' learning (Cognitive stage). This stage will help you isolate the various elements of the skill, their sequencing and the nature of the final performance. (That is, you form the concept of the skill).

2. Practice the skill. The flexible use of teaching skills in a variety of combinations required by different classroom situations cannot be learned without practice. It is best to practise the various elements in simulated or controlled conditions first. Microteaching allows the student teacher to practise specific skills under controlled conditions - practising for 5 – 10 minutes using a small class (4 – 7 students) and practicing a single skill.

3. Obtain knowledge of result or feedback. Practice is improved by receiving feedback regarding performance. Feedback takes the form of a supervisor's comments after lesson is over. (Use of videotape and audio recording to record and replay for self-evaluation).

Reflection

1. State and explain the three basic teaching functions. 2. Identify the teaching skills that support each of the functions of planning implementation and evaluation.

SESSION 2: SCHEME OF WORK

This session introduces you to preparation of a scheme of work for mathematics. The session deals with the meaning of scheme of work and a sample of a mathematics scheme of work.

Learning outcome(s)
By the end of the session, the participant will be able to identify the main features of a good mathematics scheme of work.

A **scheme of work** is a plan that defines work to be done in the classroom. It is the teacher's plan of what he or she will teach during every lesson throughout the academic year. It is a vital and useful document which teachers will need to produce. A scheme of work is a guideline that defines the structure and content of an academic course. It maps out clearly how resources (e.g. books, equipment, time) and class activities (e.g. teacher-talk, group work, practicals, discussions) and assessment strategies (e.g. tests, quizzes, homework) will be used to ensure that the learning aims and objectives of the course are met. It normally includes times and dates. It is usually an interpretation of a specification or syllabus and can be used as a guide throughout the course to monitor progress against the original plan. Schemes of work can be shared with students so that they have an overview of their course. The key parts of a "scheme of work" include: Content, Objectives or Outcomes, Methods of delivery (student and teacher activity), Assessment strategies, Resources, Other Remarks.

Good schemes of work are rare in high schools. Most teachers use only examination specifications and textbooks to guide their lesson planning, focusing on content rather than pedagogy. Good schemes should include guidance on matters such as the most effective teaching approaches, how to meet the full range of students' needs and potentials. It should also focus on what constitutes an appropriate level of challenge, relevant activities and resources that help nurture students' understanding. They should provide sufficient support for all teachers, especially new, inexperienced and non-specialist staff. They are seen as living documents, subject to regular discussion and review.

Once types of activity are considered and included in the scheme of work, lesson planning should become less problematic and, as students become familiar with the types of activity, they will learn strategies and approaches that will support them in further problem solving.

The following are two sample formats of a scheme of work and one specific completed format for 2015/2016 academic year.

Sample 1: Format for Scheme of Work

Syllabus/ Spec. Ref.	**Content and Teacher Activity**	**Student Activity (Activities to apply the content)**	**Key Skills Development and Assessment**	**Home work Deadline Date**	**Home work Back to student date**
1					
2					
…..					
	Revision and examination				

Sample 2: Format for Scheme of Work

Week Number	Objectives	Content
1		
2		
3		
……..	……	……
	Revision for test	Sample test questions and 'mock' test

SAMPLE 3: CORE MATHEMATICS - FIRST TERM 2015/2016

SAMPLE SCHEME OF WORK: SHS 2 CORE MATHS - FIRST TERM, 2015 / 2016			
S/N	**Week Ending**	**Topic**	**Subtopic**
1	18/09/15	Linear equations & word problems	• Change of subject; Simple linear equations in one variable; Solving word problems
2	25/09/15	Linear inequalities & word problems	• Solving inequalities in one variable • Solving word problems
3	02/10/15	Linear equations in two variables	• Solving simultaneous linear equations: • Elimination, substitution and graphical methods.
4	09/10/15	Surds	• Simplification of mono surds • Operations on surds (addition, subtraction, multiplication and division); Rationalization
5	16/10/15	Indices	• Laws of indices • Simplification of negative and fractional exponents • Solving equations involving indices
6	23/10/15	Logarithms	• Definition of logarithm • Relationship between indices and logarithms • Laws of logarithms; Applications of logarithms
7	30/10/15	Relations and Functions	• Types of relations and functions • Domain of a function • Rules of mapping (Linear & exponential)
8	06/11/15	Coordinate geometry	• Gradient of a line (Parallel and Perpendicular lines); • Length/magnitude of a line; Mid-point of a line
9	13/11/15	Coordinate geometry	• Equations of a straight line (general, gradient and intercept forms):- given a gradient and a point on the line, given two points on the line, and given a point and a straight line (parallel or perpendicular to the line). • Intersection of lines
10	20/11/15	Graph of Relations (linear & quadratic graphs)	• Constructing table of values of x and y for relations • Drawing of graphs (linear and quadratic graphs)
11	27/11/15	Graph of Relations (linear & quadratic graphs)	• Interpretation of quadratic graphs (nature of graph, negative and positive, increasing and decreasing parts of the graph) • Gradient (both linear and quadratic graphs) • Using quadratic graphs to solve related equations.

12	04/12/15	Variations	• Meaning of variation • Types of variation: Direct variation; Inverse variation
13	11/12/15	Variations	• Joint variation and Partial variation
14	18/12/15	Examination	

Assignment: How can this be scaled up to be like Sample 1?

Reflection

1. Take a topic in the core or elective mathematics syllabus and draw an expanded scheme of work for a period of three months.
2. State and explain three reasons why every mathematics teacher should prepare a scheme of work.
3. What is a scheme of work in mathematics? Prepare a scheme of work for SHS1 first term lasting 14 weeks.

SESSION 3: PLANNING FOR MATHEMATICS INSTRUCTION

Good planning is necessary for effective teaching and learning of mathematics. Students learn best from lessons that are interesting and carefully organized, directed by thoughtful questions and enriched by materials that develop and provide practice. This session deals with the importance of planning lessons before instructions are carried out.

Learning outcome(s)
By the end of the session, the participant will be able to plan for your mathematics instruction.

A **Lesson plan** is a clear, concise and daily record of a teacher's lessons for the week, showing what s/he plans to do daily. Research indicates that careful development of ideas, with clear explanations, careful questioning and use of teaching and learning materials is particularly important in teaching mathematical content.

The following are some important points the teacher should consider in planning for mathematics lesson:

- the objectives the students would attain
- the key questions to ask during the lesson
- the materials to be used
- the comments in the teacher's guide or teachers' handbook for teaching the topic
- the stages and/or the order in which the content will be presented.

It also involves anticipating, (predicting or envisioning) "how students might mathematically approach the instructional task(s)" you will be assigning to them during instruction (Stein, Engle, Smith, & Hughes, 2008, p.322). Anticipating students' responses includes the teacher thinking about how students are likely to interpret the tasks(s), the varied approaches they might use – correctly or incorrectly – and their underlying mathematical concepts, representations, procedures, and practices as they relate to what you expect to have learned at the end of the lesson.

Importance of Planning Lessons

Planning for mathematics instruction is important for a number of reasons:

1. It helps the teacher to establish definite objectives for a lesson. The purposes of the lesson are indicated clearly to help him/her to avoid omissions and mistakes.

2. It helps teacher to state or choose the aspect of the curriculum to be imparted to the class at a specific time frame or period.

3. It helps teacher to obtain the necessary information from books or people. Teacher tries to identify and prepare the teaching/learning materials to use during the lesson to support teaching-learning activities.

4. It helps the teacher to know what he/she will do and say throughout the lesson, to have interesting materials prepared and arranged for use, and to set out clearly what the students will do and say during the lesson.

5. It helps to develop presentation of lessons in logical stages to attain the intended learning outcomes – to present his/her work in a sensible sequence and in appropriate units of time. Planning helps the teacher to avoid unnecessary repetition.

6. It helps the teacher to ensure that a lesson begins interestingly, to maintain a good pace throughout, and to ensure that a lesson ends satisfactorily. It enables teacher to plan how to increase student involvement or to make the lesson more participatory.Planning helps in holding the students' interest and attention, whether they are working as a total class, in small groups or individually.

   It enables the teacher to make adequate preparation for specific problem areas. To plan for the use of a variety of teaching methods that will cater for the diverse group of students in the class. That is, you are able to anticipate possible learning challenges and design any necessary activities to resolve them if they arise

7. Planning creates a feeling of confidence for the teacher. He/she knows what he/she wants to do.

8. Planning also gives the teacher a way of judging how the lesson went- that is, it helps the teacher to evaluate his/her behaviours and the responses of the students during the lesson.

9. It serves as teachers' planned task to be performed at specific periods as stated on the Time Table. It makes it easier for any substitute teacher to teach in the absence of the class teacher (since the lesson plan is there to refer to)

10. In summary, summary involves a great deal of anticipation and reflection on instructional decisions with the sole aim of effective lesson delivery and supporting students' learning.

Reflection

1. State six reasons why mathematics teachers should write lesson plan.
2. State four points the teacher has to consider when planning a mathematics lesson.

SESSION 4: OBJECTIVE STATEMENT FOR A LESSON PLAN

The next decision to make after choosing a subject matter to teach is what kind of things you want your pupils to learn (what they should know and be able to do after learning a concept or a skill). Getting an objective for a lesson is one way of ensuring that efficient (mathematics) learning takes place. Without a plan which features instructional objectives, an observer is likely to misjudge the effectiveness of a teacher's classroom behaviour. In this session, we will explain what a learning objective is and the characteristics of a good lesson objective.

Learning outcome(s)
By the end of the session, the participant will be able to identify and explain the features of good mathematics lesson objectives.

Very often we are interested in describing the behaviour that is expected of our students after an instruction has been given (i.e., if learning has taken place it should lead to the acquisition of some new competency). An objective statement indicates the target set for attainment. An objective is a clear, precise statement that describes the skills and knowledge that learners should gain at the end of instruction. The statement describes an intended outcome of instruction. The statement of a clear lesson objective is key to guiding where you are going.

An objective statement should give a clear description of the behaviour expected of students at the end of instructions along with the conditions and the degree of mastery required.An objective should be stated in observable and measurable terms-of what students should be able to do by the end of the lesson. It should be **SMART**- *Specific, Measurable, Achievable, Realistic,* and *Time-bound*. To write a SMART objective, try to ask yourself the following questions and see how your objective reflects the SMART criteria.

Specific: Who is the target population? What will be accomplished?

Measurable: Is the objective quantifiable? Can it be measured? How much change is expected?

Achievable: Can the objective be accomplished in the proposed time frame with the available resources and support?

Realistic: Does the objective address the goal? Will the objective have an impact on the goal?

Time-bound: Does the objective propose a timeline when the objective will be met?

An instructional objective should state what you expect the students to know (cognitively) and be able to do (behaviourally) as a result of your lesson. It should describe how the students will show what they have learned. Instructional objectives place much emphasis on what the students will do, not upon what the teacher will do; they also indicate how learning is to be observed or evaluated. It is important to specify precisely what you want your students to learn from any given activity, and students should be told what is expected of them on completion.

There are two general types of objective, **process** and **outcome**. **Process objectives** usually focus on the activities to be completed in a specific time period. They are meant to support accountability by setting specific activities to be completed by specific dateline and explain what and when it will be completed. **Outcome objectives** refer to the expected results by the end of a specified period.

They are specific and concise statements that state who will make what change, by how much, where and when. Outcome objectives are meant to be realistic targets for an instruction and become the yardstick by which our real accomplishment will be assessed. Outcome objectives thus answer the question: Who is going to do *what*, *when*, and to *what extent?*

When selecting an instructional objective for use in teaching, use a verb which describes observable actions or actions which have observable products. e.g. to identify, to choose, to solve, to calculate, to analyse, to explain etc. Vague verbs like to know, to believe, to understand, to appreciate are not observable and should be avoided. The syllabus usually has objectives stated for the listed topics/subject matter. These are usually part of a larger set of goals. Teachers are thus free to choose appropriate materials and procedures to help them to achieve their short-term or instructional objectives.

It is not always possible to observe the thinking process of a student when he is solving a mathematical problem, but the teacher can examine the steps taken or ask the student to verbalise his or her thought process to arrive at the solution.

Examples: *By the end of the lesson, the student will be able to*:

1. (i) use the protractor to measure interior angles of several triangles and generalize that sum
   of interior angles of a triangle is 180° and

   (ii) use this rule to find the third angle of a triangle when two angles are given.
2. solve simultaneous linear equations involving two unknowns by method of substitution
3. solve quadratic equations by factoring and use the principle of $a \times b = 0$, where *a* and *b* cannot be simultaneously zero to solve quadratic equations.
4. draw the graph of a quadratic function and use the graph to solve the corresponding quadratic equation and related problems.

Reflection

1. A good lesson objective must be ‘’SMART’’. Discuss
2. Write lesson objectives for two selected contents under the following units from the SHS2 Core Mathematics syllabus:

a. Plane Geometry II (Circle theorems);
b. Trigonometry.

SESSION 5: LESSON PLANNING GUIDE

This session covers specific steps involved in lesson planning.

Learning outcome(s)
By the end of the session, the participant will be able to prepare a good lesson planning guide

Specific Steps in Planning

1. Choosing the subject-matter of the lesson;
2. Finding out what the pupils already know about the subject you have c chosen (relevant previous knowledge);
3. Specifying instructional objectives; what students should know and be able to do at the end of the lesson;

4. Devising instructional procedures (that build on what students already know including potential gaps in their knowledge) which will help you to achieve the objectives;
5. Constantly monitoring students' progress in the course of the lesson delivery.
   All these points require the teacher to think through what he/she plans to do during the lesson and having alternative strategies to use in case the initial strategy appears to be ineffective.

The Planning Guide

1. The general headings of date, class, age of students, number in class, time and duration of lesson, and subject matter – content/topic (*what am I teaching*?).

2. Particular aims or instructional objectives – exactly what do I hope the students will learn as a result of this lesson?

3. The intended structure of the lesson and how the time will be used. This should include:
   a. the teachers' work: - exposition, questioning, showing diagrams, discussion, etc.
   b. the students' work: discussion, planning and carrying out activities, completion of exercises, individual or group work etc.
   c. the order in which the work is expected to progress - accompanying diagrams might show planned work and use of other visual aids, samples of exercises set, assignments etc.
4. Materials and equipment (a) required by teacher, (b) required by students.

5. Subsequent comments (to be completed as soon as possible **after the lesson is over**), that is, post-lesson reflection:)

   a. How far the work has developed as planned

   b. Particularly good aspects of the work: where students showed interest and cooperation, useful materials, outstandingly good work, etc.

   c. Particularly bad aspects: where students showed little interest, behaviour difficulties, inadequate provision of materials, unforeseen difficulties, outstanding bad work.

   d. Assessment of the total situation, suggestions about future work, good points to be followed, and deficiencies made good, desirable modifications in original programmes.
   It should be noted that assessment of how well a lesson is, is an ongoing process occurring at various stages in the development of the lesson and not only at the end.
   Stein et al.(2008) identified 5 key practices that teachers should engage in for an effective instruction that is supportive and responsive to students' needs. These are:

      1. Anticipating possible responses from students to mathematical tasks assigned them;

      2. Monitoring students' responses to tasks during the implementation phase;

      3. Selecting particular students whose work will be presented during the whole classroom discussions. This means that you should create opportunities for students to do some work on their own, individually and/or collectively. As they work on assigned tasks, you should be going round inspecting what they are doing, what strategies are they employing? What strategies are correct or wrong? What are the underlying mathematical ideas in the various strategies being used? Going round

as students work therefore, goes beyond determining who has finished and who has not.

4. Purposefully sequencing particular students' work that would be shared with the whole class. You should think through how these works connect and the mathematical ideas, concepts, strategies, etc., that you want to bring to students' attention.

5. Helping students make connections between different works presented and the key mathematical ideas.

Reflection

1. Identify the specific steps involved in planning a mathematics lesson.
2. Outline the six major components of a mathematics lesson plan

SESSION 6: MATHEMATICS LESSON PLAN FORMAT

This last session of Unit 4 introduces you to the prescribed lesson format for your On-Campus and Off-Campus Teaching Practice exercises.

Learning outcome(s)
By the end of the session, the participant will be able to explain what goes into a mathematics lesson plan.

Prescribed Lesson Plan Format

NAME OF SCHOOL: ....

DATE	15th May 2017	16th May 2017	18th May 2017	Etc
TIME	7.00 – 8.20	10.00 – 11.20	8.40 – 10.00	
DURATION	80 min	80 min	80 min	
CLASS	Science 1A	G/Arts 1	Bus 1A	

SUBJECT: Mathematics (**Core/Elective**)

TOPIC: [Unit] e.g. Surds

SUB-TOPIC: [Content] e.g. Simplifying and operations on surds including rationalization.

REFERENCE: ...... (Use APA style. Include the **syllabus** and **textbook**(s)).........

INSTRUCTIONAL MATERIALS: ....(Indicate when it will be used in the lesson). E.g. **Worksheets** containing practice questions to be done in groups after each developmental steps.

CONTENT OUTLINE (Breakdown of sub topic/contents)

i. Simplifying (mono) surds
ii. Addition, subtraction and multiplication of surds
iii. Rationalization of surds with monomial denominators

INSTRUCTIONAL OBJECTIVES

By the end of the lesson, the student will be able to:

i. simplify surds of the form √a;
ii. carry out operations involving surds;
iii. rationalize a surd with monomial denominator.

RELATED PREVIOUS KNOWLEDGE

Students **can** prime factorize numbers and can identify perfect squares

INTRODUCTION (Estimated time … e.g., **5** minutes)

Review previous knowledge on perfect squares and prime factorization with a few questions; and/or state the topic on board, share the lesson objective with students; and/or ……….

DEVELOPMENT (Estimated time ……)

Step 1 (**20 mins**) (Use Instructional language).… e.g.

i. Explain surds as…..…
ii. Guide students to simplify surds of the form $\sqrt{a}$.
e.g. √45 = √(9×5) = 3√5.
iii. Guide students to simplify product of surds of the forms

$\sqrt{a}\times\sqrt{b}=\sqrt{ab}$; $(\sqrt{a})^2=a$; and $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$. E.g ………

iv. Distribute worksheets to students in groups to try questions on simplifying surds. (State the worksheet tasks and instructions)

Discuss worksheet results with class

Step 2: (**20 mins**) (Use Instructional language).........

i. Guide students to find sums, differences and product of surds e.g. $\sqrt{27}+\sqrt{48}-\sqrt{75}+12\sqrt{3}$.

ii. Distribute worksheets to students in groups to try questions on operations on surds. (State the worksheet tasks and instructions)

Discuss worksheet results with class

Step 3 : (**15 mins**)........... (Use Instructional language)...

i. Guide students to rationalize a surd with monomial denominator e.g. $\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$
ii. Distribute worksheets to students in groups to try questions on operations on surds. (State the worksheet tasks and instructions)

Discuss worksheet results with class

CLOSURE (SUMMARY) (Estimated time (**5 mins**))

.... (Use Instructional language).... e.g. Summarize the lesson by recapping the core points of the lesson. State the core points.

Or, call on students to identify the salient points by asking some leading questions to be answered orally.

EVALUATION (Estimated time **15 mins**)

i. Class Exercise: .... (Use Instructional language)....
e.g., Give the following questions to students to be done as class exercise/ You can refer to questions from the textbook quoting the page.

ii. Homework/Assignment............ (Use Instructional language)......
e.g., Give the following questions to student to be done as assignment or Refer to questions from the textbook quoting the page.

REMARKS (To be written after lesson has been treated or ….)

....

....

Reflection

Prepare a 40-minute lesson plan on a selected SHS mathematics topic for a class of your choice.

UNIT 5: LESSON PRESENTATION SKILLS, MOTIVATION AND CLASSROOM MANAGEMENT

Lessons which combine explanation, discussion and individual instruction require a considerable amount of verbal structuring and directing to keep classroom activities progressing smoothly. The teacher has to do a great deal of presenting – exposing students to new facts, concepts and principles, explaining difficult ideas, clarifying issues or exploring relationships. Lesson presentation skills include: Set Induction, Closure, Stimulus variation, Clarity of explanation, Use of examples and Questioning. This unit discusses lesson presentation skills and how to motivate students and manage mathematics instructions.

Learning outcome(s)

By the end of the unit, the participant will be able to:

- explain set induction and closure and the functions of set induction;
- identify and explain stimulus variation techniques;
- explain what is meant by clarity of explanation and factors that contribute to effectiveness of explanation;
- explain the guidelines for the effective use of examples in the mathematics class;
- identify effective ways of questioning and give examples of questions under Bloom's taxonomy;
- state and explain ways of motivating learners in the mathematics classroom and classroom management skills.

SESSION 1: SET INDUCTION AND CLOSURE

Set induction refers to activities which precede a learning task. It has an influence upon the outcome of that task and some instructional sets promote learning better than others. Set induction brings meaning and direction to the lesson. Set induction or pre-instructional procedure, must generally be **brief and brisk** in order to succeed in arousing the interest of the students. Set induction can also be appropriately used during the course of a lesson to:

i. begin a new unit of work;
ii. initiate a discussion;
iii. introduce an assignment;
iv. prepare for a practical session;
v. prepare for viewing a video;
vi. prepare for field excursion;
vii. introduce a guest speaker. This session introduces some forms of introducing lessons in mathematics.

Learning outcome(s)

By the end of the session, the participant will be able to:

- identify and explain the various functions of set induction,
- explain closure for mathematics instructions.

Functions of Set Induction

1. To **focus the students' attention** on what is to be learned by gaining their interest. For example, teacher introduces a lesson on:
   a. Volume of a cylinder by displaying a variety of cylindrical objects and net of a cylinder.

   b. Quadratic equation by displaying several examples of quadratic expressions/ equations such

   as $y = 3x^2 + 2x + 1,\ \ y = x^2 + 4x + 4,\ \ x^2 \pm 1 = 0,$ and $y = 2x^2 - 3x + 5.$

   c. Properties of a parallelogram by showing a number of parallelograms**.**

2. **Transition set.**
   This function provides a smooth transition from known or already covered material to new or unknown material. This is often referred to as the review of relevant previous knowledge. This is done to establish a knowledge base for the new lesson. It often takes the form of question and answer session on relevant previous knowledge for the new topic. It may also use examples from students' general knowledge to move to the new lesson by use of examples or analogies.

   Other forms are, mental work, checking homework or teacher providing a brief summary of the previous lesson which is relevant to the new lesson. For example, "We have all seen cylinders before and the funnel. We know how the funnel is made! A sector folded!"

Example 1: Standard deviation of simple data

Last week we learnt about how to find the mean of a set of numbers. We know that the mean of a set of numbers is obtained by summing all the given numbers and dividing the answer by the number of values/observations in the set. For example, to find the mean of six given numbers, we first add all the 6 numbers and then divide the sum by 6.

Today we are going to use this knowledge to calculate another important measure in statistics. Quickly write down these numbers and calculate the mean: 7, 9, 6, 13, 10, 9, 2.

Now seat in your groups and take the following sheets and then follow the instructions on the sheets. Then the worksheets are distributed and students set to work as part of the development of the lesson.

Instructions on the worksheet:

Consider the following set of scores of eight students: 3, 6, 5, 4, 7, 5, 10, 8.

1. Find the mean score.
2. Find the difference between each score and the mean: d=x-x.
3. Square each difference : d2=x-x2
4. Sum all the squared differences (deviations): d2=x-x2
5. Divide this sum by the number of scores: x-x2n
6. Find the square root of the result: x-x2n.

The answer obtained is referred to as the s**tandard deviation** which measures the square-rooted average error (deviations) between the mean of the dataset and each observation in the dataset. As a measure of deviations away from the mean, the smaller the value, the better since we prefer deviations to be very small to give indication that the model is a good one.

Example 2: Sequences and Series

We learnt about sets of numbers that follow a certain pattern – (i) increasing or decreasing by a fixed number for any two consecutive numbers in the given pattern; or (ii) consecutive terms form a constant quotient (ratio). These sets of numbers are called sequences and the numbers in each set are called the terms of the sequence.

Now answer the following questions:

i. What name is given to the type of sequence in which consecutive terms differ by a fixed number?
ii. What name is given to the type of sequence in which there is a constant quotient when you divide a term (other than the first term) by the previous term?

3. **To provide structure or framework for the lesson**.
   Teachers can influence students' behaviour best when they are told in advance what is expected of them. This is called "the expectancy functions of teachers" (DeCecco 1968). Gage and Berliner (1975) refer to it as "advance organizers"

   Set induction attempts to create an organizing framework for the body of the lesson. That is, providing the gist of the lesson or an outline of the main ideas of lesson or activities that will be done. A sufficient set induction gives adequate preparation so that while engaged on a learning activity the student is able to come near to the stated instructional objectives.

   Example 1: We shall be looking at different types of triangles and find out that no matter the type, and size, when we sum up the three angles we will get the same answer.

   Example 2: Today we will learn more about the first type of sequence (Linear Sequence or Arithmetic Progression (AP)). We will be finding the sum of any given number of terms in a linear sequence. We will engage in an activity to develop a formula to be used to find the sum of first n-terms of an A. P.

4. To give **meaning** to a new concept or principle. The introduction to an activity can also contain guides or cues which will be helpful to students in understanding the lesson. Appropriate use of examples and analogous situations can help students to understand abstract ideas**.**

   Example: When you have students membership in clubs (Maths, Science, Drama), some students can be in all three clubs or two clubs or in only one club. This makes calculating the total number of students involved quite tricky/special. We shall learn about the use of Venn diagram to solve this kind of problem in today's lesson.

5. Giving a **historical background** related to the topic.
   For instance, History/ story on Gauss to introduce summation of series, or Pythagoras to introduce lesson on lengths of right angled triangle, or history of the Sieve of Eratosthenes to introduce prime numbers.

   Example: Here is a story about a brilliant young school boy who gave a big surprise to his teacher.

One day the teacher came to class and gave a very challenging task to the class (Thinking it was enough to engage the students for quite a long time). The task: "*Add all the counting numbers from 1 to 100*". It took a young boy called Carl Fredrick Gauss no time to give the correct answer as 5050.

So surprised, the teacher asked Gauss to explain:

Do you know what Gauss did?. We shall learn about that in today's lesson.

6. Stating and explaining the **objectives**. It is better to overstate the objective than to understate it. This often times includes the importance (utility) of learning about the new lesson. By the end of the lesson, I expect all of you to be able to explain the steps involved in solving a quadratic equation by factorization…. and to ….

7. The simple task of stating the **topic** (writing the topic on the board).

8. Presenting (**posing) a problem situation** that can be solved with ideas from the lesson. (Real life
problems are often most appropriate). For instance, you can pose a story problem and give to students to look at while *you give an excuse to attend to an urgent issue with Headmaster in the office.*

The Problems:

Example 1: Simultaneous linear equations in 2 variables

A student bought 3 notebooks and 1 pen for GH¢35.00. Her friend also bought 2 notebooks and 2 pens from the same shop for GH¢30.00. What is the cost of one pen?

Example 2: Trigonometry

A man resting on a branch of a tree 5 metres from the ground observes a bird on top of a tall vertical TV pole 15 metres away from the tree on which the man is. The pole is 25 metres tall.

i. *Find the angle through which the man should look in order to see the bird.*
ii. *If he attempts to shoot the bird with a catapult, what is the shortest distance the stone will have to travel to reach the bird's position at the top of the pole?*

Example 3: Lowest Common Multiple

Alia empties the dustbin every 12 days, mows the grass every 18 days and pays the bills every 15 days. Today he did all three. How long will it be before he has another day like today?

Closure

Closure is a complement of set induction. It draws attention to the end of a specific learning sequence, or of an entire lesson by focusing on what has been learned. It needs to be carefully planned, allowing adequate time to initiate closure before the lesson is due to end.

Effective closure reinforces what has been learned by reviewing the key points of a lesson and relating them to other materials the students have already learned.

The main objective of closure is to help the students retain the important points presented in the lesson, thus increasing the possibility that they will be able to recall and use that information at another time.

It is sometimes used during the course of a lesson:

- to end a discussion by calling on a student(s) to summarize the major points covered,
- to end a practical work (discovery) by calling on different students to list the steps carried out, the results obtained and the conclusion drawn.
- to follow up a homework /assignment reviewed in class by using praise and encouragement, example, "I know that was quite difficult but you did your best. I am pleased with the output. Keep it up." This is **social closure**. It gives students a sense of achievement and very useful for difficult learning tasks.

Closure takes any of the following forms:

1. Teacher summarizes the main points.
2. Teacher asks students to summarize the main points.
3. Teacher consolidates major points and ideas during lesson before moving on to new subtopics.
4. Teacher gives encouragement and praise.

Reflection

1. Explain five functions of set induction.
2. What is the main purpose of set induction?
3. What is the main purpose of closure for a mathematics lesson?
4. Explain three ways closure can be done.

SESSION 2: STIMULUS VARIATION

Stimulus variation refers to those teacher actions, sometimes planned and sometimes spontaneous, that develops and maintains a high level of attention on the part of the students, during the course of a lesson. This skill is to arouse students' attention in order to focus it upon the content of the lesson. One effective way of doing this is to make the content itself interesting. However, an interesting content can be made tedious by the manner of presentation.

Animated behaviour on the part of the teacher stimulates the attending behaviour of students and enhances learning (Rosenshine, 1970). The skill of stimulus variation is based on learning theory which indicates that uniformity of the perceived environment tends to lead students into mental inactivity, while changes in the perceived environment attract their attention and stimulate mental activity. In this session you will learn about various stimulus variation techniques.

Learning outcome(s)

By the end of the session, the participant will be able to explain the various stimulus variation techniques.

Stimulus Variation Techniques

Stimulus variation techniques include: teacher movements; focusing behaviour; changes in speech pattern; changing interactions and shifting sensory channels.

1. **Teacher movements**
   This can have an important effect on students' behaviours. For example, physical movement from one part of the classroom to the other causes students to focus attention directly on the teacher during presentation. But random nervous movements – pacing up and down can irritate and interrupt the communication. Walking and talking should be avoided.

2. **Focusing behaviour**
   Focusing is the teacher's way of intentionally directing students' attention. This can be by verbal statements, specific gestures or movements or some combination of the two.

   a. *Verbal focusing* involves emphasis of particular words, statement or directions. For example, "Look at the diagonals, all coming from one vertex" "Watch the powers of the variable in each expression"
   b. *Gestural focusing* consists of eye movements, facial expression and movements of head, arms and body (Argyle 1970). Gestures are important means of communication between teacher and students. They are used to gain attention and to indicate emotions
      i. *Gaining attention* – use of eye contact with the entire class, use of a position to indicate an object, turning the body in the direction of an object, clapping the hand to gain attention.
      ii. *Indicating feeling or emotion.* Smiling, frowning, raising eyebrows, nodding the head to give encouragement.
   c. *Verbal – Gestural* Focusing combines both. Sometimes this is for emphasis example, point to a diagram and say look at this diagram.

3. **Changes in speech pattern**
   Variations in quality, expressiveness, tone and rate of speech can increase animation. For example, a change in the teacher's rate, volume or tone of speech can increase the students' attention. Planned silence or pausing can also be most effective in capturing attention by contrasting sound with silence. It creates a suspense or expectation (a sudden pause in the middle of a sentence). About 3 seconds is the enough for a pause. Nervous teachers are often afraid to allow for a pause. They rush to fill any pause with extra questions or statements. Experienced teachers often pause after asking a question and may pause again to prompt a student to continue if the student shows that he can extend an answer. The teacher may combine the pause with a smile and a nod to indicate encouragement.

4. **Changing interactions**
   There are three main types of interaction.

a. *Teacher/ group interactions*

Here, teacher lectures or demonstrates to a whole class and so it is a teacher – centred (less participatory) approach. Questions are directed to the group as a whole and not to individuals specifically.

b. *Teacher – student interaction*

This is teacher – directed. Teacher directs questions to a specific student in order to promote student exposition and discussion. For example, a student has completed an activity and teacher has a set of questions based on the activity for that student.

c. *Student – student interaction*. This can take any of the following forms:

i. Class discussion in which teacher plays a management role, redirecting students' questions to other students for comments and clarification.
ii. A class working in small groups on an activity and they discuss among themselves. Teacher only plays a management role.
iii. Students go to the board to show the steps they used to solve a problem

5. Shifting Sensory Channels

Students process information by means of the senses – sight, touch, smell, taste and hearing. Studies have shown that students' abilities to process information can be significantly increased by appealing to sight and sound.

a) Listen – look – listen: This is verbal explanation followed by diagram (object/picture) and then verbal questions.

b) Look – listen – look (students observe a picture/object then answer questions from teacher and then observe again)

c) Listen/ look; touch/look – teacher provides instruction using a diagram, followed by providing the real materials/ objects to perform the activity.

Reflection

1. What is meant by stimulus variation in lesson presentation?
2. Explain four stimulus variation techniques.
3. What are the main forms of focusing behaviour?
4. Identify and explain the three types of changing interactions.

SESSION 3: CLARITY OF EXPLANATION

According to Bellack et al (1996), to explain is to relate to an object, event, actions or state of affairs to some other object, events, action or state of affairs and a principle or to show a relation between principles or generalizations. It is much difficult to explain than to give a factual report. Lack of clarity is a major barrier to the use of explanation. Clarity of presentation is an aspect of behaviour which has considerable influence on the effectiveness of classroom teaching. This session deals with factors that contribute to the effectiveness of explanation.

Learning outcome(s)

By the end of the session, the participant will be able to explain the factors that contribute the effective use of explanation during mathematics instruction.

Factors that Contribute to Effectiveness of making an Explanation

1. Continuity

i. **Sequence of discourse**: Teacher should make the connections between the various points dealt with in a lesson very obvious to students. This involves purposeful selection and sequencing of tasks in ways that build on each other and enable students to have a clear idea of the key mathematical ideas, principles or skills to be learned (Stein, Engle, Smith, & Hughes, 2008). Diversions from a central theme should be minimal and where a diversion is necessary it should be made clear that it is a diversion. If this is not so, students may find it difficult to know the points to focus on.

ii. **Fluency**: The use of easily intelligible grammatical sentences helps students to understand a teacher's explanation. Success in this depends on the teacher's mastery of subject –matter and his social confidence in class and on a careful advance planning of lesson. Lack of fluency often makes teacher to leave most of his sentences unfinished or to interrupt his own sentences in order to reformulate them. Teacher should frame his questions in such a way that it is possible for students to answer them without additional help. In order to do so, it is important that you plan the questions you want to ask in terms of the key mathematical idea, the appropriateness of language and the order to pose them. Give your students adequate time to think and suggest answers. Give prompting information or rephrase your question only if an adequate answer is not forthcoming.

2. Simplicity:

i. **Avoid a grammatical complexity.** Do not include too much information in one sentence. Keep sentences short. Communicate complex relationships by visual means. The use of diagrams, tables and models simplifies relationships that are difficult to explain by speech.

ii. **Vocabulary**. Use simple language within the students' normal vocabulary for effective communication. Introduce specialist terms only when they are necessary for mastering a concept. Always attempt to use language within the experience of the students. In mathematics, language serves as a key mediational tool to convey what about a mathematical object we expect students to know and be able to do. We use language to name mathematical objects (a triangle, quadratic equation, etc.) and to convey what is valued (legitimating criteria) with respect to the object of learning (i.e. the instructional objective), (Adler & Ronda, 2015).

3. **Explicitness.**

It is often wrong to assume that there is more common ground between the teacher and the students. This often makes teachers to be less explicit. The use of phrases like "of course", "you know '' are signs of teacher's vagueness. Others are "a little, "some", "many" "small", "large" instead of using exact figures.

A clear explanation depends on the teacher's ability to:

a. identify the components that are to be related e.g., objects, events, processes, and generalizations.

b. identify the relationship between the components e.g. causal, justifying, and interpreting.

A teacher's failure to do this leads to a confused presentation.

The extent to which a teacher makes an explanation explicit is related to students' attainment. Teacher should make use of explaining links. These include phrases like, because, as a result of, why, therefore, so that, in other that, by, etc. For example, this is a rectangle because it has 4 sides and each angle is 90°. This is a trapezium because it has 4 sides and a pair of opposite sides parallel. This is a parallelogram because it has 4 sides and opposite sides are parallel and equal.

Reflection

1. Explain three factors that contribute to making effective explanation in class.

SESSION 4: USE OF EXAMPLES

The use of examples is a basic teaching skill commonly used to bring a mathematical idea into students' focus. Examples therefore serve as instantiations of a particular aspect of a mathematical idea, concept or a mathematical structure (Mason, Stephens, & Watson, 2009). That is, we use examples to bring into focus an aspect of a mathematical idea, concept, or mathematical structure so that they can make some form of generalization from the class of examples presented. For effective teaching, teacher should be able to **use** examples and **seek** examples from students in such a way as to help students to comprehend the new concepts. This would require that you choose examples that enable students to make meaning and appreciate mathematical structures and make the necessary abstractions and generalizations. This session deals with inductive and deductive use of examples and guidelines for effective use of examples.

Learning outcome(s)

By the end of the session, the participant will be able to:

- distinguish between inductive and deductive use of examples;
- explain the guidelines for effective use of examples.

Similarity and Contrast

According to Adler and Ronda (2015):

Two necessary aspects for a sequence of examples that constitute a basis for generalization are **similarity and contrast**. We draw on the work of Watson and Mason (2006), who attend to variation amidst invariance in mathematics, and on variation theorists (e.g. Marton & Tsui, 2004), to categorise three different patterns of variation: similarity (S), contrast (C) and fusion (F). ….Focusing on what something is through a set of similar examples brings possibilities for generalizing that which is invariant. Similarity on its own, however, does not draw attention to the boundaries around a concept, and so to what it is not. **Contrasting** examples that bring attention to a different class also make available opportunity for generality. Moreover, further generality is possible through fusion—when more than one aspect of an object of learning is simultaneously varying/invariant across an example set. p.4.

Think of the following set of examples:

A	B	C
1. $y = 2x + 3$	4) $y = 2x - 3$	7) $y = 2x + 3$
2. $y = 2x + 5$	5) $y = 2x - 5$	8) $y = 4x + 3$
3. $y = 2x + 7$	6) $y = 2x - 7$	9) $y = -2x + 3$

a. In class, discuss what is variant or invariant in each example set?
b. Identify examples showing similarity in each set of examples, A, B, and C.
c. Identify contrasting examples in example sets A and C
d. What generalization can students make from these example sets?

Concepts allow us to organise and store similar pieces of information efficiently. Once formed, they eliminate our need to treat each new piece of knowledge as a separate category. Martorella

(1972) conceives concepts as hooks on which we hang new experiences. Concepts organise our knowledge structure and keep it from becoming unwieldy and dysfunctional. Concepts help to speed up and simplify communication. If the concept is formed by both the teacher and the students, communication is easy. There will be no need to explain details.

It follows that effective teaching of new concepts depends on the teacher's ability to;

- make use of students' existing concepts
- systematically organizing instructional activities (use of examples) in ways that enable students to understand mathematical concepts and
- use relatively concrete information in attempting to communicate relatively abstract ideas.

Teacher should focus on situations and ideas well within students' experience and understanding. This enables students to perceive common features and thus to abstract generalizations. For instance, when a statement is ambiguous or not understood by students, the teacher should clarify it by the use of examples drawn from students experience and present level of understanding. The teacher needs to know much about his students so that he can give relevant examples.

Inductive and Deductive uses of Examples

The use of examples in teaching can be done inductively or deductively. Inductive approach is where the teacher starts with examples and makes an inference or generalization based on these examples. Deductive approach is where the teacher states the generalization first and then applies it to a number of examples. None is known to be superior to the other. The effectiveness of the deductive approach is that the initial generalization, even when not understood, focuses the attention of students on the aspect of the examples the teacher wants them to know. The effectiveness of the inductive approach lies in helping students to acquire skills of looking for order (in an apparently pattern-less set of data),

Guidelines for Effective use of Examples

1. Start with the simplest examples that will achieve your goals, and then to more complex examples.

2. Start with examples relevant to the students' experience and level of knowledge.

3. Relate examples to the principle, idea or generalization being taught. The point of using examples is to illustrate, clarify or substantiate a principle, generalization, idea or rule. This is likely to be achieved if different examples are used showing similarity, contrast, and fusion of more than one idea. You should enable students to make decisions about what is similar (same) and what is varying, that is, what is varying and what is not varying (variant/invariant) regarding given examples sets in order for them to build towards generalization.

4. Check to see whether you have accomplished your objective by asking students to give examples which illustrate the point you were trying to make. If students cannot give examples, then further clarification is necessary involving simpler examples or analysing the generalization into several parts.

Reflection

1. Why do teachers use similarity and contrast in their use of examples in mathematics classes?
2. Distinguish between inductive and deductive use of examples.
3. Explain four guidelines for effective use of examples.
4. Write four examples illustrating the idea of a) similarity and b) contrast.

SESSION 5: QUESTIONING

Questioning is probably the most important activity teachers engage in. It is a major part of their teaching repertoire. Stephens (1912) found that high school teachers ask almost 400 questions during an average school day. Other studies reinforced this, Floyd (1960) gave an average of 348 questions; Moyer (1966) gave 180 questions for a science lesson. Bellack et al (1966) observed that "the core of the teaching sequence found in many classrooms is a teacher's question, a student's response and more often than not a teacher's reaction to that response". In this session you will learn about Bloom's taxonomy and profile dimension.

Learning outcome(s)

By the end of the session, the participant will be able to classify questions under Bloom's taxonomy and profile dimensions.

The kind of questions the teacher asks will reveal to the students the kind of thinking which is expected of them. Many teachers resort to memory –type questions drawn directly from the textbook. Studies have shown that different types of questions stimulate different kinds of thinking (Taba, Levine and Elzey, (1964). Teachers should therefore be conscious of the purpose of their questions.

Questioning and questioning techniques are crucial to a good lesson plan. Remember you are trying to stimulate thought in your students as they are active learners in the class. Consider the level of questions you are asking. The questions included in your lesson plan should be upper level questions. Higher order questions are defined as anything more complex than knowledge level. They require careful thought in advance of the class. Listing upper-level questions in your plans shows that you have given them appropriate consideration. Distribute participation throughout the class. Higher-order questions are designed to make students think, and reflect about what they are learning. They are to strengthen students' ability and communication skills. They are difficult to create extemporaneously in front of the class. How? and Why? can be upper-level questions when connected to a response given by a student.

Classifying Questions

One of the best known classifications is Bloom's taxonomy of educational objectives (1956). There are six levels each requiring questions to provoke different thought processes. These are knowledge, comprehension, application, analysis, synthesis and evaluation.

Knowledge

The purpose of knowledge questions is to determine whether students remember certain specific facts. Recall is observable when a student states a specific fact or gives information in much the same from as presented by the teacher or a textbook. Recall answers can easily be judged as right or wrong.

Knowledge of specific facts

- Which of the following number(s) is/are not rational? $\frac{0}{3}, \frac{5}{0}, \frac{15}{4}, \sqrt{5}.$
- State the addition algorithm for rational numbers $\frac{a}{b}+\frac{c}{d}.$
- What is $64 \div 8$?

- What are the factors of 6?
- State the binomial theorem.

Knowledge of terminology

- Two sets having the same number of elements are said to be _______ sets.
- 15! = _______
- Two sets that do not have any members in common are said to be _______

Teachers tend to over use this category of questions. Knowledge questions assess only a superficial understanding. Much of what is memorized is rapidly forgotten.

The words, *what, define, when, recall, recognize/identify, name, list, etc.* are often used in framing knowledge questions.

Comprehension

These questions are used when teacher wants students to organise facts in such a way as to make some sense of them. They allow students to select facts that are pertinent in order to describe, compare or contrast.

Knowledge of concepts

- In what way does the set of whole numbers differ from the set of natural numbers?
- Suppose A and B are two acute angles p° and q° respectively. A and B are complementary if and only if …….
- Distinguish between odd and even numbers [abundant and deficient numbers]
- Explain the difference between a prism and a pyramid.

Knowledge of principles, rules and generalizations

- If three fractions have denominators that are relatively prime, the least common denominator is equal to ………………
- If the decimal point in a number is moved four places to the right, we are …….

Comprehension questions also allow students to translate ideas from one medium to another, e.g. to interpret information presented in graphs or tables, i.e., *From the table find the number of people who scored less than X marks.*

Note that information necessary to answer comprehension questions should already have been provided. Commonly used words are, *describe, compare, put in your own words, explain, contrast etc.*

Application

Application questions are asked to encourage students to apply information they have learned in order to find an answer to a problem. They require students to apply a rule or a process to a problem to determine the correct answer.

Ability to solve routine problems

- Which of the following numbers, expressed in base eight numeration, is both prime and odd?
  13_{eight}, 23_{eight}, 34_{eight}, 17_{eight}, 31_{eight}
- Given $\log_b 2 = 0.693$ and $\log_b 3 = 1.099$, find $\log_b 12$.
- A piece of wire 36 metres long is bent into the form of a right angled triangle. If one of the legs is 12 metres long, find the length of the other leg. [9 metres]
- Find the area of a circle with radius 35 cm and $\pi = 3.14$.
- Use the binomial theorem to find the expansion for $(2x-1)^7$.
  Commonly used words are *apply, classify, use, choose, employ, given an example* etc.

Ability to make comparison

- The difference in the circumference between the larger and smaller of two balls is 3cm. Which of the following is the best estimate of the difference in their diameters? ....
- Two different saving plans are available to you. At the bank, your savings will earn 5% compounded quarterly. The Savings and Loans Association pays 5.25% compounded annually. Where should you place your money to earn the maximum interest?

Ability to analyse data

- Five tests are given to Kofi's class. Each test has a value of 25 points. Kofi's average for the first four tests is 15. What is the lowest score he can get on the fifth test to have an average of at least 16?
- Joe wanted 6 pairs of socks. Store A sold them at 2 pairs for 1.25 Ghana cedis; the same brand sold in store B at 3 pairs for 1.98 Ghana cedis. To be economical, what should Joe do?
- On the same axes, sketch the graphs of y = sin x and y = cos(½ x) as x varies from 0 to 2π radians. Determine from the graphs the quadrant in which sin x – cos(½ x) is always positive.

Ability to recognize patterns and symmetries

- The last digit in 4^{10} is ……
- In an election, 400 people vote to choose one of five candidates. The candidate with the most votes is the winner. What is the smallest number of votes the winner could receive?

Analysis

These questions are meant to help students to remember and organise information and also to analyse the information for underlying reasons such as cause and effect. Students are made to analyse information to reach conclusions or to find evidence.

- Why does the sine curve not go beyond -1and1.
- What is the largest rational number which is the sum of two rational numbers each having a numerator of 1 and a whole for its denominator?
- Given that $p = r^2 - t^2$, where p is a prime number, find t if (i) $r = 7$ (ii) $r = 157$, (iii) $r = 58$.
- Determine the number of lines obtained by joining n distinct points in the plane, no three of which are collinear.
- What conclusion can you draw about the graphs of quadratic functions?
- The length of the diagonal of a square is (x + y). Find its area.
- Find the number of diagonals in a convex polygon with (i) 5 sides (ii) 25 sides (iii) n sides.

- Prove that for every positive integer $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is an integer.
- What is wrong with the following proof for "*any two numbers are equal*?

ii. Given that $a+b=c$

From (i) we obtain the equation (2): $a=c-b$ and (3): $b=c-a$

Multiply (2) by (– b) and (3) by (– a) to get (4) $b^2-bc=-ab$ and (5) $a^2-ac=-ab$.

From (4) and (5) we have (6): $b^2-bc=a^2-ac$

Add $\frac{c^2}{4}$ to each side of (6) to get (7): $b^2-bc+\frac{c^2}{4}=a^2-ac+\frac{c^2}{4}$.

Take the square root of both sides of (7) to get (8): $b-\frac{c}{2}=a-\frac{c}{2}$

Add $\frac{c}{2}$ to each side to get $b = a$.

Analysis questions require critical thinking from students. Commonly used words are, *why, what factors? , draw conclusion, determine evidence, support, substantiate.*

Syntheses

They are to help students to form relationships and put things together in new or original ways. They help students to develop creative abilities, they test a thorough knowledge/understanding of a subject and often require students to predict, to solve problem. Synthesis differ from application in that synthesis questions do not require answer to problem that have a single correct answer but allow a variety of creative answers. Commonly used words and phrases are; *predict, produce, write, develop, what would happen if ……….?*

Evaluation

These questions help students to choose among alternatives by judging which best fits a stated value. They do not have a single correct answer but require students to judge the merit (and demerit) of an idea, a solution to a problem, or an aesthetic work. Commonly used words are; *judge , assess, decide, justify.*

Questions can be subdivided according to the cognitive thought required. Lower-order questions require the student to recall information. Higher-order questions require students to manipulate information for some purpose. Higher order thinking often changes the form of information in order to…. compare or contrast, or explain or summarise or analyse or synthesise or evaluate. That is, application, analysis, synthesis and evaluation questions fall under higher –order thinking. Knowledge and comprehension obviously fall in to the lower –order thinking category.

Profile dimensions

A 'dimension' is a psychological unit for describing a particular learning behaviour. More than one dimension constitutes a profile of dimensions. A specific objective may be stated with an action verb as follows: The student will be able to <u>describe</u>….. etc. Being able to "describe" something after the instruction has been completed means that the student has acquired "knowledge". Being able to explain, summarize, give examples, etc. means that the student has understood the lesson taught.

Similarly, being able to develop, plan, solve problems, construct, etc. means that the students can “apply” the knowledge acquired in some new context. Each of the specific objectives in the SHS syllabus contains an “action verb” that describes the behaviour the students will be able to demonstrate after the instruction. “Knowledge”, “Application”, etc. are dimensions that should be the prime focus of teaching and learning in schools. It has been realized unfortunately that schools still teach the low ability thinking skills of knowledge and understanding and ignore the higher ability thinking skills. Instruction in most cases has tended to stress knowledge acquisition to the detriment of the higher ability behaviours such as application, analysis, etc. The persistence of this situation in the school system means that students will only do well on recall items and questions and perform poorly on questions that require higher ability thinking skills such as application of mathematical principles and problem solving. Students should be encouraged to apply their knowledge, develop analytical thinking skills, develop plans, generate new and creative ideas and solutions, and use their knowledge in a variety of ways to solve mathematical problems while still in school. Each action verb indicates the underlying profile dimension of each particular specific objective.

The two profile dimensions that have been specified for teaching, learning and testing at the SHS level are: Knowledge and Understanding (KU) with weighting of 30% and Application of knowledge (AK), including application, analysis, synthesis, and evaluation with weighting 70%.

The percentage weights show the relative emphasis that the teacher should give in the teaching, learning and testing processes at Senior High School.

Reflection

1. Plan a short micro lesson of 5 – 10 minutes concentrating on the introduction and arrange to present it to a few of your colleagues.

SESSION 6: MOTIVATION AND CLASSROOM MANAGEMENT

To strengthen the ways student involvement and motivation are promoted and supported in mathematics class the teacher has to give students tasks that require them to think about mathematical relationships and concepts. The teacher also has to provide feedback to students that promotes further thinking and improved understanding. Finally, teacher has to allow opportunities for students to be an authority in mathematics. Teachers need to praise their students when they succeed in challenging problems or projects. Teachers should <u>not</u> overemphasize testing or grades. Doing so can cause students to lose interest in the concepts they are learning and encourage them to focus only on their scores. In this session we will learn about motivation of students and management of mathematics classes.

Learning outcome(s)

By the end of the session, the participant will be able to identify ways by which teachers can motivate students to learn mathematics.

Motivation in the Mathematics Classroom

A strong motivation to master mathematics can make a big difference. Intense motivation helps students overcome disappointing mistakes, expend effort to figure out complex mathematics problems, sacrifice time to improve mathematics skills and remember mathematics learnt for a longer period of time. Using the correct strategies, you can create an environment that incites motivation for mathematics.

In order to teach mathematics, you need to know mathematics. The content mathematics courses are usually learnt in college or in the university. The topics learnt in these contents courses need to be blended and to see the interconnections. In addition to knowing mathematics, you need to know about the age-characteristics of the students you teach and about teaching. The challenge is to stimulate students to want to learn something in mathematics.

Motivating students to succeed in their mathematics studies can stump even the most experienced teachers at times but there are **techniques you can use to increase their motivation** as much as possible. These techniques are:

1. Be passionate about what you're teaching. If you have a favourite math concept or if one problem really challenged you, point that out to students. Let your own enthusiasm for the subject show in your attitude towards teaching.

2. Find out what most interests your students. If your students are interested in a specific sport or to current events, tie in any applicable math concepts to that sport or to events in the newspaper.

3. Discuss the history behind the math. Often, explaining when a problem was solved and who solved it can help students relate to the people behind the math process, as well as giving them mathematical role models that they can look up to.
4. Explain how the information they learn in class can help them in real life. For example, you may want to introduce them to different interesting fields in which they will need this information, such as forensics, nuclear physics or even cooking (when you are teaching fractions and proportions).

5. Teach through discovery learning. Instead of teaching a concept and having students apply it to several problems, give students a problem and challenge them to solve it. When they are engaged in the problem-solving process, they will be more interested in the concept.

6. Give students the freedom to choose as much as possible. If students can choose which concept they will learn next, they'll be more invested in understanding it well. If students can choose the project they would like to do to illustrate a concept, they will feel as if they have more control over their learning.

7. Let them understand that making mistakes is normal and encouraged in your class. Encourage them to see mistakes as opportunities to learn better and therefore urge them to share their thinking in class freely. Getting an answer is not an indication of a student's inability to succeed in mathematics.

Tips and Warnings

1. Praise your students when they succeed in challenging problems or projects. That is, praise should be used sparingly and for effort and not just for obtaining a good score (Dweck, 2010). Feedback you give should target what students can do in order to improve upon their learning.

2. Do not overemphasize testing or grades. Doing so can cause students to lose interest in the concepts they are learning and encourage them to focus only on their scores. Test for conceptual understanding and not speed.

Here is one way of stimulating students to learn what they need to and what you want them to learn. Instead of directly teaching students how to do long multiplication such as 23×436 and to bore them with routine procedure, you may introduce some other approach.
We can express 23 as a sum of powers of 2 by listing the powers of 2 below 23 until the next power will exceed 23. That is, 1, 2, 4, 8, 16.

Then we successively double the second factor, 436 to match from the first power of 2 to each ensuing one as shown.

23	**436**	
1	436	
2	872	Double 436 to match 1st power of 2
4	1744	Double 872 to match 2nd power of 2
8	3488	Double 1744 to match 3rd power of 2
16	6976	Double 3488 to match 4th power of 2

But from the table, $23 = 16 + 4 + 2 + 1$, so we pick the corresponding values (doubles) below 436. That is, $6976 + 1744 + 872 + 436 = 10{,}028$, which is 23×436. The double, 3488 has not been included in the sum because its corresponding power of 2 (3rd power, 8) does not form one of the addends of 23.

Students are likely to ask if there is an easier way to multiply. They are motivated and this gives you the opportunity to show the standard algorithm for multiplication. Here you have not told students about the need to learn the algorithm, they have asked for it. There is motivation here. This approach may also invite some students to learn the new approach or research it to find out that this is a method used by Egyptians for multiplication.

Some Factors Related with High Achievement

The following are some factors related with *high achievements* and some suggested ways teachers can organize teaching to encourage *high achievements* in the classroom.

1. **Attention**
   When teaching is designed so that student active participation in the learning activity is demanded, attention is usually higher. For example, asking all students to hold up a card giving the sum of two numbers, (or, the derivative of a given function, or the gradient of a line) is better than calling only one student to give the answer.

Teachers' strategies for selecting students to participate during discussions influence attention and active participation. Thus, a teacher who calls only on volunteers to answer questions will find that other students will stop paying attention. For successful lessons, it is desirable to keep the lesson moving at an even pace, to try to prevent interruptions, and to select activities that will involve and stimulate the students.

2. Initiative

Learning is most efficient when students can identify points where they need help and then obtain it. Thus, willingness to initiate contact with the teacher promotes learning. Teachers should let students know when they could and should demonstrate initiative, thus encouraging this behaviour.

3. Understanding

Students who see a task as worthwhile and understand clearly how to complete it are more likely to continue to work at it. Thus, teachers need to further students' understanding by making clear how any particular work should be done and what the reasons are for doing it.

4. Success

Students' long-term achievement is positively related to their success in daily classroom tasks. Therefore,

c. Assignments should be matched to students' abilities.
c. Work in progress should be monitored and prompt feedback provided. (If a student makes a mistake and does not realize it, this may affect his/her performance on the rest of the task.

Grouping for Teaching and Learning

There are three basic patterns of grouping for teaching and learning in a mathematics classroom. These are (i) Whole class with teacher guidance; (ii) Small group, either with teacher guidance or with student leaders; and (iii) Individuals working independently.

1. Mathematics teachers are advised to use *large-group* pattern for teaching and learning if:

a. the topic is one that can be presented to all students at approximately the same point in time, that is, if all students have the same pre-requisites for understanding the initial presentation.
b. pupils will need continuous guidance from the teacher in order to attain the knowledge, skill and understanding.

2. *Small-group* work can mean that students work on a content focused on their needs and at the same time learn to work together to solve problems. The Three-Part Lesson format and the Think-Pair-Share strategy are more often organized for small groups. Use small-groups for teaching and learning if:

a. students can benefit from student - student interaction with less teacher guidance
b. activities involve a few students at a time.
c. they want to foster co-operative learning skills.

3. Mathematics teachers are advised to use *individual teaching* and learning approach if:

a. students can follow a sequence or conduct an activity on their own
b. the focus is on individual practice for mastery

Setting up a safe and friendly environment in the mathematics classroom.

The following are some ways of achieving good classroom management.

a. organizing group work, peer teaching and presentations;

b. respecting the opinions of students;
c. avoiding confrontations with students;
d. having fun with students by using exciting review games to help them learn;
e. identifying ways of motivating students to learn mathematics;
f. creating opportunities for students to ask questions;
g. rewarding students appropriately;
h. providing them with interesting but challenging tasks;
i. creating opportunities for students to discuss among themselves –collaboration;
j. teaching from known to unknown;
k. encouraging students to believe in themselves that they are capable of doing mathematics.

Reflection

1. State and explain three techniques you will employ in your mathematics class to increase your students' level of motivation.
2. Explain three factors that could promote high students achievement in a mathematics class.
3. Identify any two ways the mathematics teacher can use problems to motivate the learner

UNIT 6: TEACHING SELECTED SHS TOPICS

The Pythagorean Theorem is a very useful theorem in the study of mathematics. It is applied in solving problems on Bearings and Coordinate Geometry. Logical Reasoning aids people's ability to analyse and to make valid arguments. In this unit, we will learn how to guide learners to teach Pythagorean Theorem, Bearings, and Co-ordinate Geometry. We will also learn about how to teach Logical Reasoning.

Learning outcome(s)
By the end of the unit, the participant will be able to:

- derive and use the Pythagoras Theorem;
- find the bearing of one point from another;
- find the equation of a straight line and the magnitude of a line segment;
- write mathematical statements and their negations;
- write compound statements using 'and', 'or' and 'implication'
- form chain rule and determine the validity of conclusions.

SESSION 1: TEACHING THE PYTHAGOREAN THEOREM
The triangle has long been known to be a very "practical" geometric figure. The rafters of a house are fastened together in the shape of triangles in order to help make the framework rigid. Triangles have been used in the construction of bridges, buildings and others. The ancient Greeks also did a lot of work with triangles, particularly with right-angled triangles. An important relationship between the longest side of a right-angled triangle and its two shorter sides was attributed to Pythagoras. This relationship is known as the Pythagorean Theorem. In this session, we will discuss how to guide high school students to derive the Pythagorean Theorem and how to use the Theorem to solve real life problems.

Learning outcome(s)
By the end of the session, the participant will be able to derive the Pythagorean theorem and to use the Theorem to solve related problems.

The Pythagorean Theorem
The Pythagorean Theorem states that the number of unit squares found in the square built on the hypotenuse (the longest side) of a right-angled triangle is the same as the sum of the number of unit squares found in the squares built on the two shorter sides of the triangle.
To show this we raise squares on the sides of a right-angled triangle ABC and divide them into unit squares. Guide students to identify the hypotenuse as the longest side and the side which faces the right angle (90°). Try the following practical activity.

Figure 6.1(a)

We determine the number of unit squares on the longest side that is, 25, and those on the shorter sides as 16 and 9 as shown in the next diagram. We notice that the sum of the unit squares on two shorter sides is equal to the number of unit squares on the longest side; that is 16 + 9 = 25.

Figure 6.1(b).

This is repeated for right-angled triangles with sides of various measures; say 6 cm, 8 cm and 10 cm. In each case we notice that the sum of the unit squares on the two shorter sides is equal to the number of unit squares on the longest side of the right-angled triangle.

The Pythagorean Theorem in Algebraic Form.

In the algebraic form, the theorem is stated as follows:

If for a right-angled triangle, with '*a*' and '*b*' as the measures of the two shorter sides and '*c*' is the measure of the hypotenuse (longest side), then $a^2 + b^2 = c^2$.

We can prove the theorem intuitively as follows:

We consider a right-angled triangle with the two shorter sides of measures '*a*' and '*b*' and hypotenuse of measure '*c*'.

Figure 6.2 (i) Figure 6.2 (ii)

Guide your students to observe that in figure 6.2 (i) and Figure 6.2. (ii) the same-size square has been partitioned in two ways. In Figure 6.2(i) we find a square and four triangles all of the same size. Let students observe also that in Figure 6.2 (ii) we find two smaller squares and again four triangles all of the same size. Therefore the square in the Figure 6.2 (i) must have the same area as the sum of the squares in figure 6.2 (ii). We can write this relationship as $a^2 + b^2 = c^2$, where c^2 is the area of the square in Figure 6.2 (i) and $a^2 + b^2$ is the sum of the areas of the two smaller squares in Figure 6.2 (ii).

Guide students to notice that the converse of the Pythagorean Theorem is also true and is of great use. The converse of Pythagorean Theorem states that if the measures *a*, *b*, and *c* of the sides of a triangle are such that $a^2 + b^2 = c^2$, then the triangle is a right-angled triangle with '*a*' and '*b*' the measures of the shorter sides and '*c*' the measure of the hypotenuse. Thus, a triangle whose sides measure 3 cm, 4 cm, and 5 cm is a right-angled triangle since $3^2 + 4^2 = 5^2$.

We can find out whether triangles whose sides are multiples of the measures of the $3 - 4 - 5$ triangle are also right-angled triangles. For example, if the measures of the sides of a triangle are 6, 8 and 10, then it is a right-angled triangle because $6^2 + 8^2 = 10^2$. We notice that any positive integral multiple of the $3 - 4 - 5$ triangle is a right-angled triangle.

Reflection

1. How does the sum of the areas of the squares on the shorter sides of a right-angled triangle compare to the area of the square on the hypotenuse? Make a generalization.
2. Describe how you would guide your students to use graph paper to decide if the Pythagorean Theorem is true for right-angled triangles with the following shorter sides.

a. 5cm and 12cm;

b. 8 cm and 15 cm.

3. Explain how you would guide your students to calculate the lengths of the space diagonals in the cuboids below.

(Answers: 13 m   38.7 cm)

4.   Explain how you would guide your students to calculate the lengths of requires lines in the diagrams below.

(Answers: EF = 9.4 cm   EC = 7.8 cm)

SESSION 2:   TEACHING BEARINGS

This session deals with how you can guide your students to solve problems on bearing.

Learning outcome(s)
By the end of the session, the participant will be able to describe the steps you would take your students through to solve problems on bearing.

Bearing of a Point from Another
Guide the students to interpret bearing as direction of one point from another. For example, in finding the bearing of a point A from O, we can use the four cardinal directions North, South, East and West.

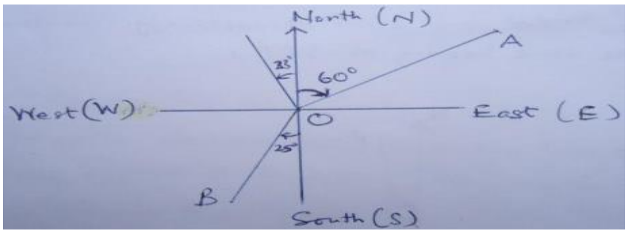


Figure 6.3

We refer to a turning of 60° from the North (N) towards the East (E) as a bearing of N 60° E. Thus, we can describe the bearing of A from O as N 60° E and B from O as S 25° W. When bearings are measured in this way, they are always measured from the North or South (never from the East or West). Take students through more activities of measuring bearings making reference to the North or South.

Introduce students to the alternative way of giving bearings or directions which is becoming more popular. That is, by reckoning North to be zero and measuring angles from the North in a clockwise direction. In this method North is taken as direction of reference and the angle is given in three digits. For example, 008° is written for 8° from the North, and 032° for 32° from the North. In this system East is written 090°, South 180° and West 270°. The bearings considered earlier (N 60° E) and (S 25°W) may also be written 060° and 205° respectively. Similarly, N 23° W is the same bearing as 337°.

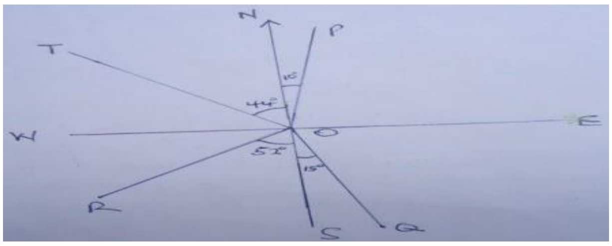


Figure 6.4

Task your students to find bearings of the points P, Q, R and T from point O in Figure 6.4. Check to see if the students get the results as follows:
The bearing of P from O is 010°,
the bearing of Q from O is 165° (that is, 180° - 15°),
the bearing of R from O is 232° (that is, 180° + 52°),
and the bearing of T from O is 316° (that is, 360° - 44°).

Reverse Bearing
Suppose we are told that the bearing of a point A from a point O is 060°. This can be illustrated as in Figure 6.5.

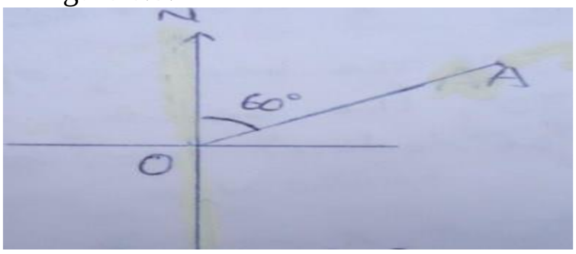


Figure 6.5

Now guide your students to deduce the reverse bearing that is, the bearing of O from A by erecting the cardinal points at A and taking the bearing of O from A.

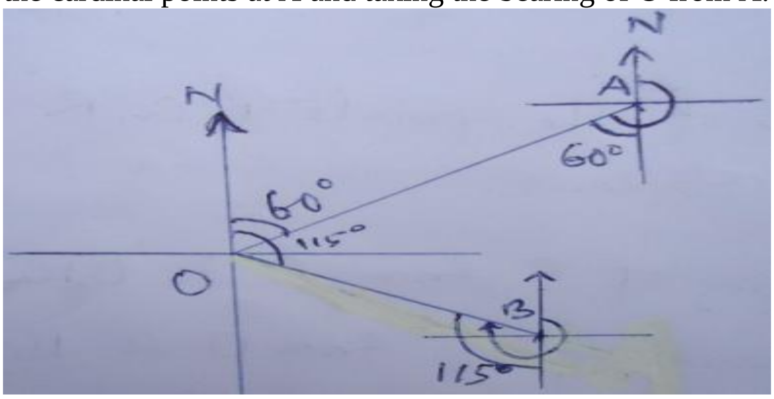


Figure 6.6

The bearing of O from A is given by (180° + 60°) = 240°
Guide students to determine that, if the bearing of B from O is 115°, then the bearing of O from B is $180° + 115° = 295°$.
Give students more similar problems to solve and conclude that, *if the bearing of a point P from point O is x°, then the bearing of point O from P (that is, the reverse bearing) is given by* $(180 + x)°$ for $0° \leq x \leq 180°$.

Also, if the bearing of point C from point O is 230°, the bearing of point O from point C can be deduced from Figure 6.7 as shown.

Figure 6.7

From Figure 6.7,

$$\angle GCO = \angle HOC \quad \text{(alternate angles)}$$
$$= 230° - 180° = 50°$$

Therefore, the bearing of point O from point C is 50°

Also, if the bearing of point D from point O is 332°, the bearing of point O from D can be deduced from Figure 6.7. That is,

$$\angle KDO = \angle HOD \quad \text{(alternate angles)}$$
$$= 332° - 180° = 152°.$$

The bearing of point O from point D, therefore, is 152°

Take students through more similar examples and guide them to observe and conclude that, *if the bearing of a point P from a point O is y°, then the bearing of point O from point P (that is, the reverse bearing) is given by* $(y - 180)°$ *for* $180° \leq y \leq 360°$.

From the examples above, guide students to deduce the reverse bearings generally as follows: if Y is *z*° from X, then the reverse bearing, that is, X from Y is:

a. $(180 + z)°$ for $0° \leq z \leq 180°$;
b. $(z - 180)°$ for $180° \leq z \leq 360°$.

Distance – Bearing Form
Guide your students to represent bearing in the distance-bearing form. Explain to them, for example, that a journey along a particular bearing has distance and direction and hence it can be represented as (r, θ), where ***r* km** is the distance and θ the bearing.
Consider a journey of ***r* km** on a bearing of θ (Figure 6.8)

Suppose the easterly and northerly components of the journey is ***x*** and ***y*** respectively. Guide students to apply Pythagorean Theorem to find the length of the journey, which forms the hypotenuse of the resulting right-angled triangle formed. That is, the length of the journey, ***r* km**, is given by $r = \sqrt{x^2 + y^2}$

If θ is the bearing of the journey, then $\tan\theta = \dfrac{x}{y}$, which gives $\theta = \tan^{-1}\left(\dfrac{x}{y}\right)$.

Question: Suppose a car travels 6 km east and then 8 km north. Find the distance and direction of the car from the starting point.

Solution:
Guide students to represent the information on a diagram as shown.
Suppose the final destination of the car is point P and the bearing of P from O is θ, the journey can be represented as (r, θ).

Thus, $r = \sqrt{6^2 + 8^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100} = 10$

$\angle OPK = \theta = \tan^{-1}\left(\dfrac{6}{8}\right)$

$\Rightarrow \quad \tan^{-1}(0.75) = 37°$

Therefore, the car travelled 10 km on a bearing of 037°. i.e. (10 km, 037°)

Reflection

1. Explain how you would guide SHS1 students to find:
   a) the bearings of A, B, C and D from O (the origin of a Cartesian plane)
   b) the reverse bearings of O from A, B, C and D.
   (Diagram not drawn to scale)

2. Show and explain how you would guide an SHS1 student to solve the following problem: *A man walks from town A 3km east to town B and then 4km south to town C. Find the distance and bearing of town C from town A.*

SESSION 3: TEACHING COORDINATE GEOMETRY

The coordinate system consists of a horizontal number line, the x-axis, and a vertical number line, the y-axis. The intersection of the axes is called the **origin**. The x- and y-axes divide the coordinate plane into four **quadrants**, identified as quadrants I, II, III and IV. Points in the coordinate plane are determined by ordered pairs of numbers. The first number in an ordered pair, is called the ***x*-coordinate** or the **abscissa;** the second number is called the ***y*-coordinate** or the **ordinate**. In this session we will discuss how to guide participants to find the gradient and equation of a straight line and how to find the distance between two points.

Learning outcome(s)

By the end of the session, the participant will be able to find the:

i. gradient of a straight line;
ii. equation of a straight line;
iii. magnitude of a line segment.

Gradient of a straight line

The gradient or slope of a straight line or a line segment is the degree of its steepness, or the rate at which it rises or falls. Consider the examples in Figure 6.9.

In figure 6.9 the gradient of the line OA is $\frac{3}{5}$, and the gradient of the line OB is $\frac{2}{-6} = -\frac{1}{3}$.
Guide students to read or interpret the gradients of drawn lines from graphs. That is, for line OA, the gradient $\frac{3}{5}$ is from the interpretation that for every vertical (upward) move of 3 (steps), there is a corresponding horizontal (rightward) move of 5 (steps). For line OB, the gradient $\frac{2}{-6}$ is interpreted as "for every vertical (upward) move of 2 steps there is a corresponding horizontal (leftward) move of 6 steps.

Can you now write the gradients of line OC and OD?

I hope your answers are $\frac{-2}{-5} = \frac{2}{5}$ for line OC and $\frac{-3}{3} = -1$ for line OD.
That is, for every vertical (downward) move of 2 (steps), there is a corresponding horizontal (leftward) move of 5 (steps) for line OC.

Similarly, gradient of line OD equals $\frac{-3}{3} = -1$ is interpreted as for every vertical (downward) move of 3 (steps), there is a corresponding horizontal (rightward) move of 3 (steps).

Check the readings carefully and make your own rule for interpreting given gradients.

Now guide students to derive the formula for finding gradient of a line given the coordinates of two points on the line.
Generally, if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on a straight line, the slope is determined by the change in the vertical distance $(y_2 - y_1)$ divided by the change in the horizontal distance $(x_2 - x_1)$ (see Figure 6.10).

Figure 6.10

If we denote the gradient of the line by *m*, then *m* is the ratio of the change along *y*, the vertical axis to the change in *x*, the horizontal axis. That is, $m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\text{rise}}{\text{run}}$.

Using algebra, the gradient, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $x_2 - x_1 \neq 0$.
There are two special cases of gradient that we have to note:

If a line is horizontal, then $y_2 = y_1$ and $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0}{x_2 - x_1} = 0$.
The gradient of a horizontal line is zero.

We observe also that, if a line is vertical, then $x_2 = x_1$ and $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_2 - y_1}{0}$.
Since the denominator of *m* is zero, the gradient is **undefined**.
Thus, the gradient of a vertical line is undefined. This is why the definition of gradient includes the condition that $x_2 - x_1 \neq 0$.

Equation of a Straight Line
A relationship that can be represented by a straight line graph is called a linear relationship. A linear relationship can be represented by an equation which may be written in different forms. We will consider two of the forms.

Equation of a straight line given its gradient and a point on it.
Suppose we are given the gradient, ***m***, of a straight line and a point $P_1(x_1, y_1)$ on it.
Guide students to consider an arbitrary point $P(x, y)$ on the line as in figure 6.11.

From Figure 6.11. the gradient of the line is $m = \dfrac{y - y_1}{x - x_1}$.
Students to cross multiply to get, $y - y_1 = m(x - x_1)$

The result, $y - y_1 = m(x - x_1)$, is the equation of a line with gradient ***m*** and passing through the point (x_1, y_1).

Now ask students to use the formula to write the equation of a line with gradient $\dfrac{2}{3}$ and passing through $(6, 5)$.

Solution

It is given that $m = \frac{2}{3},\ x_1 = 6,\ y_1 = 5$.

Substituting these values into $y - y_1 = m(x - x_1)$ gives $y - 5 = \frac{2}{3}(x - 6)$.

Expanding and clearing fractions, we have $3y - 15 = 2x - 12$

Simplifying gives $2x - 3y + 3 = 0$.

This final equation, $2x - 3y + 3 = 0$, is of the form $ax + by + c = 0$ which is the general form of writing the equation of a line.

Equation of a straight line passing through two given points.

Suppose a straight line passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Now guide students to consider an arbitrary point $P(x, y)$ on the straight line as shown in figure 6.12.

From Figure 6.12, we find the gradient of the line using the two given points P_1 and P_2.

Gradient, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Then, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

This may also be written as $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

or $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

Ask students to check the algebraic manipulations.
Now ask your students to use the formula to solve the following problem.

Question:
Find the equation of the line passing through (2, 3) and (4, 7).

Solution

The given values are $x_1 = 2, y_1 = 3;$ and $x_2 = 4, y_2 = 7$

Students substitute the given values into the formula $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

That is, $y - 3 = \frac{7 - 3}{4 - 2}(x - 2)$

This simplifies to $y-3=\frac{4}{2}(x-2)$

$$\Rightarrow y-3=2(x-2)$$

Expanding, students get $y-3=2x-4$

Simplifying gives $y=2x-1$

Or, in the general form as $2x-y-1=0$.

Magnitude of a Line Segment

To find the magnitude of a line segment is to find the distance between the end points of the line segment.

Guide students to derive the formula for finding the magnitude of a line joining two given points.

Suppose we are given the line segment with the endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Triangle PQR is a right-angled triangle. So using Pythagorean Theorem, we have

$$|PQ|^2=|PR|^2+|RQ|^2$$

$$=(x_2-x_1)^2+(y_2-y_1)^2$$

$$|PQ|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Thus, the distance, ***d***, between two points (x_1, y_1) and (x_2, y_2) is given by $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Reflection

1. Describe how you would guide SHS students to find the gradient and equation of the line segment with end points $A(-1, 4)$ and $B(3, -1)$.   2. Show and explain clearly how you would guide SHS students to fine the magnitude of the line segment whose endpoints are $A(3, -4)$ and $B(-2, 8)$

SESSION 4: LOGICAL REASONING - STATEMENTS AND THEIR NEGATIONS

This session is to help you and your SHS students to appreciate the importance of logical and abstract thinking in solving mathematics and real life problems. The session deals with mathematics statements and their negation.

Learning outcome(s)

By the end of the session, the participant will be able to:

- identify and form true or false statements;
- form the negation of simple statements.

Statements

Brainstorm with students to bring out the meaning of statements. Ask them to write down any sentence and use the responses to define statements as "Any group of words that gives a fact about something is referred to as a *statement.*" For example,

1. John came to school on Monday.
2. The goat was allowed to graze on the field.
3. Ama is taller than Doe.

These are statements. They may either be true or false. A sentence which is either true or false but never both is a statement.

Guide students to note that each of the three preceding sentences may or may not be true. However, they give a fact about somebody or something. The sentence, *Ho is the regional capital of Central Region of Ghana* is not true. However, it is a statement.

Now consider the following sentences. (a) Who is dancing? (b) What is going wrong? (c) Send the pen back. (d) Clean the chalkboard.

Explain to students that these are not statements because they are neither true nor false. The first two, (a) and (b), are questions. The third and fourth, (c) and (d), are commands or instructions or orders.

Letters such as P, Q, R, S and T are often used to represent statements. For example,

1. $P : 6 \in$ {multiple of 3}
2. Q: Accra is the capital of Gambia.

The first statement is read "*P is the statement 6 is a multiple of* 3". This is a true statement.

The second is read "*Q is the statement Accra is the capital of Gambia*". This is a false statement.

Let us consider the statement. P: Desire is a brilliant student.

Guide students to notice from the statement that (a) there are students in general;

(b) there are brilliant students; and (c) there is a particular student called Desire.

Thus, the set of all students forms the Universal set, i.e. U = {students}. Brilliant students form a subset, B = {brilliant students} and Desire is a particular element; d = Desire in set B.

Now guide your students to represent the statement on the Venn diagram as shown.

Now ask your students to seat in small groups and try the following question.

Question 1: Use Venn diagram to represent each of the following statements.

a. P: 5 is a prime number.
b. Q: Kafui is a medical student.
c. R: 9 is a perfect square but 7 is not.

Discuss group results with the class.

Solution

(a) U = {numbers}

P = {prime numbers}

$5 \in P$

(b) U = {students}

M = {medical students}

•k = Kafui; $k \in M$

(c) U ={numbers}

S = {perfect squares}

$9 \in P$

$7 \notin P$

Now introduce problems involving two or more subsets of a universal set to your students. Guide students to solve the following questions.

Question 2:

Explain how you would guide the SHS3 student to represent the following statements on a Venn diagram:

a. T: Teachers are important workers
b. P: Mathematicians are clever people. Q: Eli is a Mathematician. R: Doe is clever but not a mathematician. S: Kesse is not a clever person

Solution

Guide students identify 'workers' to form the universal set, and the subsets are 'important workers' and 'teachers' from the statement in a), T: Teachers are important workers.

That is, U = {workers}, I = {Important workers}

and T = {Teachers}, $T \subset I$.
Students represent the information on a Venn diagram as shown.

Guide them to identify 'people' as the bigger set containing 'clever people' from the second statement, *P: Mathematicians are clever people.* Then Mathematicians form a subset of clever people. Eli, Doe and Kesse are elements in the identified subsets.

U = {people},

M = {mathematicians}

C = {clever people},

e = Eli; $e \in M$

d = Doe; $d \in C, \quad d \notin M$

k = Kesse; $k \notin c, \quad k \notin M$

Now put students into cooperative groups and ask them to solve the following questions.

Question 3:

Illustrate the following statements on a Venn diagram.

P: All my colleagues are hard-working

Q: Hardworking people are rich

R: Mawunya is hard-working but not my colleague

S: Nelson is my colleague

T: Tommy is not rich.

Discuss the group results with the class.

Solution

U = {people}, R = {rich people}

H = {hardworking people}, C = {colleagues}

m = Mawunya, $m \in H, \ m \notin C$

n = Nelson, $n \in C$

t = Tommy, $t \notin R$

Negation of Statements

Negation of a statement is usually formed by using the operator ***'not'*** denoted by " ~ ".

For example suppose we have the statement. P: All birds fly.

Then the statements "Not all birds fly" is the **negation** of P. The negation of P is denoted by ~ P. That is, ~P: Not all birds fly. Or, ~P: It is not the case; or, ~P: It is not true that all birds fly.

We notice that for any true statement, the negation is false and for any false statement, the negation is true. If you negate a negation of a statement you get the original statement. That is, ~ ~P = P . That is, Negation of ~ P is P.

Guide your students to write the negation of each of the following statements.

a. P: $\frac{1}{2} \times 10 > 4$. (b) Q: A triangle is a polygon. (c) R: $7 \leq y \leq 10$.

Solution

a. $\sim P: \frac{1}{2} \times 10 > 4$, or $\sim$ $P: \frac{1}{2} \times 10 \leq 4$

b. $\sim Q$: a triangle is not a polygon

b. $\sim R$; $y < 7$, or $y > 10$

Now ask students to try the following question:

Question 4:

Negate each of the following statements and state whether the negation is true or false

a. P: All prime numbers are odd (b) R: $17 \equiv 2 \bmod 3$ (c) S: $7 \in \{\text{factors of } 56\}$

Solution

(a) ~P : Not all prime numbers are odd. (True)

(b) ~ R: $17 \neq 2 \bmod 3$; (False)

(c) ~S: $7 \notin \{\text{factors of } 56\}$; (False)

Guide students to recall the fact that if $a \in A$ then $a \notin A'$. We can see that complement of a set is the negation of the given set. That is the negation of a statement about a set A is a statement about A'. We can therefore use Venn diagram to illustrate the negation of given statements.

Ask students to solve the following question.

Question 5:

Illustrate the negation of each of the following statements on a Venn diagram.

a. P: Sebu is a member of the mathematics club

c. R: Mathematics students do well in English language.

d. T: P is not divisible by 5.

Solution

(a) ~ P: Sebu is not a member of the mathematics club.

U = {club members} or U = {people}, M = {members of math club}

s = Sebu

(b) ~R = Maths students do well in English language.

U = {students}, E = {students who do well in English language}

M = {mathematics students}

(c) ~T: P is divisible by 5

U = {numbers}, F = {numbers divisible by 5}

$p \in F$

Reflection

Explain how you would guide your SHS3 students to solve the following questions.

1. Determine whether each of the following statements is true or false.

a. A rhombus is a quadrilateral. b) A trapezium is a rectangle.

c) 8 is a prime number d) $23 \div 3 < 9$

e) 13 is a factor of 104 f). log 25 = 2log5

2. Illustrate each of the following statements on a Venn diagram

a) P: Religious people are trustworthy. b) $R: 19 \in \{\text{prime number}\}$

c) S: 8 is a factor of 32. d) T: Factors of 12 are also factors of 36.

e) P: Poor people are lazy. f) Q: Lazy people are poor.

3. Represent the following statements on a Venn diagram.

a) P: Factors of 42 are also factors of 84. Q: 12 is a factor 84.
R: 7 is a factor of 42. S: 28 is not a factor of 42 but it is a factor of 84.

b) P All SHS2 boys are obedient. Q: Badu is an obedient boy.
R: Addo is in SHS2. S: Gabu is not obedient.

4. Represent the negation of each of the following statements on a Venn diagram. State whether the negation is true or false.

a) P: All counting numbers are divisible by 1. b) Q: Some integers are not odd.

c) P: $3 < 5$. d) Q: $5 \leq x < 9$.

e) R: $2 < y < 6$. f) R: Hardworking students pass their examinations.

SESSION 5: COMPOUND STATEMENTS

So far, we have dealt with simple statements such as P: $a < 3$; Q: $15 \times 3 \in \{\text{odd numbers}\}$ and R: A rhombus is a parallelogram. We can combine two or more simple statements using operators "*and*", *or*, *if...then*. When two or more simple statements are combined into a single one, we call such a statement *Compound Statement*. In this session, we will discuss with you how to combine simple statements to form compound statements of the various forms.

Learning outcome(s)
By the end of the session, the participant will be able to construct conjunction, disjunction and implications from simple statements.

Conjunction (*and*)

Suppose we have the statements P: $x \geq 1$ and Q: $x \leq 5$. We can combine the two statements using the operator '*and*' to obtain $P\ and\ Q$. That is, "$x \geq 1$ and $x \leq 5$". This combination is written in the simplified form as $1 \leq x \leq 5$. Thus, $P\ and\ Q$: $1 \leq x \leq 5$ is called the conjunction of "$P\ and\ Q$".

Symbolically, the conjunction of $P\ and\ Q$ is denoted by '$P \wedge Q$'. The compound statement thus becomes $P \wedge Q : 1 \leq x \leq 5$.

Now guide your students to combine given simple statements using the operator "and".

Question 6:

Combine each of the following pairs of simple statements using '*and*'

(a) P: Kojo is a good boy. Q: Kojo is in the Mathematics club.

(b) P: $x < 9$. Q: $x \geq 5$.

(c) P: 18 is divisible by 3. Q: 18 is divisible by 9.

Solution

a. $P \wedge Q$: Kojo is a good boy and Kojo is in the mathematics club.
Or $P \wedge Q$: Kojo is a good boy and he is in the mathematics club.

c. $P \wedge Q$: $x < 9$ and $x \geq 5$ i.e. $P \wedge Q$: $5 \leq x < 9$.

d. $P \wedge Q$:18 is divisible by 3 and 9.

Now guide students to consider the following statements.

P: $2 < 7$. Q: $9 > 2 + 5$.

Then the compound statement is $P \wedge Q$: $2 < 7$ and $9 > 2 + 5$.

Let them notice that the statement P is a true statement and the statement Q is also true.

Let them also observe that the compound statement $P \wedge Q$ is also true.

Similarly, $P : 5 \in \{\text{odd numbers}\}$ and $Q : 5 \in \{\text{prime numbers}\}$ are true statements. Then the compound statement $P \wedge Q$: $5 \in \{\text{odd numbers}\}$ and $5 \in \{\text{prime numbers}\}$ is also true. That is, 5 is an odd number and also a prime number.

Guide them to draw the conclusion that:

In general, if both P and Q are both true statements, then the compound statement $P \wedge Q$ is also a true statement.

Discuss the use of sets and Venn diagram to interpret the compound statement, $P \wedge Q$.

Suppose *P* and *Q* are statements and *A* and *B* are subsets of a universal set. Suppose *P* is a statement about set *A* and *Q* is a statement about the set B. Then the compound statement $P \wedge Q$ is a statement about the subset $A \cap B$, the intersection of *A* and *B*.

From the example, $P : 5 \in \{\text{odd numbers}\}$ and $Q : 5 \in \{\text{prime numbers}\}$, we can have the sets $A = \{\text{odd numbers}\}$ and $B = \{\text{prime numbers}\}$ as subsets of $U = \{\text{natural numbers}\}$.

Notice from the diagram that $5 \in A \cap B$ indicating that 5 is both an odd number and a prime number.

Now ask students to try the following question.

Question 7

Form a compound statement from each of the following pairs of statements using “and”. State whether the compound statement is true or false.

a. P: 7 is divisible by 3. Q: 7 is an odd number.
b. P: $2 < x < 4$. Q: x is even
c. P: $\log_2 64 = 6$. Q: $\tan 45^{\circ} = \frac{\sqrt{2}}{2}$

Discuss the solution with the class.

Solution

(a) $P \wedge Q$: 7 is divisible by 3 and 7 is an odd number. Since statement *P* is false the compound statement $P \wedge Q$ is also false.

Using sets, we have U = {Whole numbers}. A = {numbers divisible by 3} and B = {odd numbers}

Notice that $7 \notin A \cap B$. Hence, the statement about $A \cap B$, that is, $P \wedge Q$ is not true. It is not true that 7 is both an odd number and a number divisible by 3.

(b) $P \wedge Q : 2 < x < 4$ and *x* is even.

This statement is false because if *x* is a number between 2 and 4 then it cannot be even. Observe that the only whole number between 2 and 4 is 3 which is not even. Also, if *x* is even, then it cannot be a number between 2 and 4.

(c) $P \wedge Q$: $\log_2 64 = 6$ and $\tan 45^\circ = \frac{\sqrt{2}}{2}$. This statement is false since Q: $\tan 45^\circ = \frac{\sqrt{2}}{2}$ is false.

Lead students to conclude that:

In general, the compound statement $P \wedge Q$ is false if one of the statements P and Q is false.

Disjunction ("Or")

Explain to your students that compound statements that are formed by using the operators '*or*', and "*either…or*" are called *disjunction*. The compound statement '*P* or *Q*' is symbolically written "$P \vee Q$".

We now use specific examples to illustrate.

(i) Suppose *P*: 2 is a prime number and *Q*: 2 is even. We form the compound statement P or Q, thus: $P \vee Q$: 2 is a prime number or 2 is an even number.

The statement $P \vee Q$ is a true statement. (Notice that statement P is true and statement Q is also true).

(ii) Consider the statements, $P: x < 7$ and Q: $x > 9$

The compound statement P or Q is given as $P \vee Q: x < 7$ or $x > 9$

We can interpret $P \vee Q$ as either the statement *P* is true or statement *Q* is true or both statements *P* and *Q* are true. Observe that this compound statement will be true even if one of the statements *P* or *Q* is false. That is, if $x < 7$ is true but $x > 9$ is false, the statement $P \vee Q$ will still be a true statement.

Guide students to draw the conclusion:

In general, the compound statement $P \vee Q$ is true if at least one of the simple statements P or Q is true.

That is, if (1) *P* is true but *Q* is false, $P \vee Q$ is true.

(2) *P* is false but *Q* is true, $P \vee Q$ is true.

(3) *P* is true and *Q* is true, $P \vee Q$ is true.

Now given the statements: *P*: 8 is an odd number and *Q*: 8 is divisible by 3,

then the compound statement *P* or *Q* is given by

$P \vee Q$: 8 is an odd number or 8 is divisible by 3.

Clearly, this is a false statement because it is not true that 8 is an odd number and it is also not true that 8 is divisible by 3. The two simple statements *P* and *Q* are false. Students conclude that:

In general if both statements P and Q are false, then the compound statement $P \vee Q$ is also false.

In terms of sets and Venn diagram, we can say that if P is a statement about set A and Q is a statement about set B then the compound statement $P \vee Q$ is a statement about $A \cup B$.

Now give students the following question to try.

Question 8

Form compound statements from P and Q using the operator '*Or*'. Determine whether $P \vee Q$ is true or false.

(a) P: Prime numbers are divisible by 3. Q: 28 is a multiple of 7.

(b) P: A rectangle is a quadrilateral. Q: A rhombus is a quadrilateral.

(c) P: $\{\text{natural numbers}\} \subset \{\text{integers}\}$. Q: $\{\text{real numbers}\} \subset \{\text{rational numbers}\}$

(d) P: Whole numbers are divisible by 5. Q: $2^3 = 6$.

Discuss the solution with the students.

Solution

a. $P \vee Q$: Prime number are divisible by 3 or 28 is a multiple 7.
The statement P: prime numbers are divisible by 3 is false but statement Q: 28 is a multiple of 7 is true.

Hence the compound statement $P \vee Q$ is true.

b. $P \vee Q$: A rectangle is a quadrilateral or a rhombus is a quadrilateral.
The two simple statements P and Q are true.

Hence the compound statement $P \vee Q$ is true.

c. $P \vee Q$: $\{\text{natural numbers}\} \subset \{\text{integers}\}$ or $\{\text{real numbers}\} \subset \{\text{rational numbers}\}$.
The statement P is true but statement Q is false. That is, it is not true that the set of real numbers is a subset of the set of rational numbers. However, the compound statement $P \vee Q$ is true because one of the statements is true.

d. $P \vee Q$: Either whole numbers are divisible by 5 or $2^3 = 6$.
The two statements P and Q are false. Hence the compound statement $P \vee Q$ is also false.

Implication

One other way of combining simple statements is by the use of "*if …then*…". Let us consider the following statements:

$P : x \in \{\text{factors of } 8\}$. $Q : x \in \{\text{factors of } 24\}$.

We can combine these statements using the '*if* …*then*… operator. That is,

If $x \in \{\text{factors of } 8\}$, then $x \in \{\text{factors of } 24\}$. This means if *x* is a factor of 8, then it is also a factor of 24.

The compound statement 'if *P* then Q' is called an *implication*.

Symbolically, we write $P \Rightarrow Q$ and read it as "*P* implies *Q*" or "if *P* then *Q*". Implications are also referred to as *conditional statements*.

The reverse $Q \Leftarrow P$ read as *Q* is implied by *P* means the same as $P \Rightarrow Q$.

Diagrammatically, we have $U = \{\text{whole numbers}\}$, $A = \{\text{factors of } 8\}$, $B = \{\text{factors of } 24\}$.

Notice that $A \subset B$. That is, all factors of 8 are factor of 24. Thus, the implication $P \Rightarrow Q$: If $x \in \{\text{factors of } 8\}$, then $x \in \{\text{factors of } 24\}$ is a true statement.

Now let students solve the following question.

Question 9

Form an implication for each pair of statements. State whether $P \Rightarrow Q$ is true. Illustrate with a Venn diagram.

a. *P*: *x* is a Ghanaian. Q: *x* speaks a Ghanaian language.

b. *P*: $x \in \{2, 4, 6, 8\}$. Q: *x* is divisible by 2

c. *P*: $3 < y < 7$. Q: $y \in \{\text{odd numbers}\}$.

d. *P*: a number is divisible by 3. Q: a number is divisible by 6.

Solution

a. $P \Rightarrow Q$: If *x* is a Ghanaian then *x* speaks a Ghanaian language.
Diagrammatically, we have

$U = \{\text{people}\}$ $A = \{\text{Ghanaians}\}$, $B = \{\text{speak a Ghanaian language}\}$

From the Venn diagram $B \not\subset A$.

This implies that not all Ghanaians speak a Ghanaian language.

Hence $P \Rightarrow Q$ is false.

b. $P \Rightarrow Q$: If $x \in \{2, 4, 6, 8\}$ then *x* is divisible by 2.
Using Venn diagram, we have

$U = \{\text{whole numbers}\}$, $A = \{2, 4, 6, 8\}$, $B = \{\text{divisible by } 2\}$

From the diagram, $A \subset B$ implying that all numbers in set *A* are divisible by 2. Hence, $P \Rightarrow Q$ is true.

(c) $P \Rightarrow Q$: If 3 < y < 7, then $y \in \{\text{odd numbers}\}$.

Using Venn diagram, we have $U = \{\text{whole numbers}\}$, $A = \{4, 5, 6\}$, $B = \{\text{odd numbers}\}$

Notice from the diagram that $A \not\subset B$. That is, not all elements of *A* are members of *B*.

Hence, $P \Rightarrow Q$ is false.

(d) $P \Rightarrow Q$: If a number is divisible by 3, then it is divisible by 6.

Using Venn diagram, we have $U = \{\text{natural numbers}\}$, $A = \{\text{numbers divisible by } 3\}$ $B = \{\text{numbers divisible by } 6\}$

From the diagram, we note that all numbers divisible by 6 are also divisible by 3, since $B \subset A$.

However, the implication $P \Rightarrow Q$ means that all numbers divisible by 3 are also divisible by 6.

That is, $A \subset B$. Note that 9 is divisible by 3 but it is not divisible by 6 and $9 \in A$ but $9 \notin B$.

Hence, $P \Rightarrow Q$ is false.

Students finally draw the conclusion that:

In general, the implication $P \Rightarrow Q$ *is false only when P is true and Q is false.*

Converses of implication

We will now discuss the reverse implication. Given the implication $P \Rightarrow Q$, the reverse implication $Q \Leftarrow P$ is called the **converse** of the implication $P \Rightarrow Q$.

For example, suppose *x* is a composite number and Q: *x* has more than 2 factors. Then we form the conditional statement:

$P \Rightarrow Q$: If *x* is a composite number, then it has more than 2 factors.

The converse of $P \Rightarrow Q$ is $Q \Leftarrow P$: if *x* has more than 2 factors then it is a composite number.

Notice from this particular case that the implication $P \Rightarrow Q$ and its converse $Q \Leftarrow P$ are both true statements.

If $P \Rightarrow Q$ *and its converse* $Q \Leftarrow P$ *are both true we say that statements P and Q are equivalent. Symbolically we write* $Q \Leftrightarrow P$ for equivalent statements.

Question 10

Write the implication and its converse for each pair of statements. State whether the converse is true. a) P: $2x + 4 = 8$; Q: $x = 2$ (b) P: $\frac{x}{2} + 3 < 7\frac{1}{2}$; Q: $x < 9$

Solution

(a) $P \Rightarrow Q$: If $2x + 4 = 8$, then $x = 2$. $Q \Leftarrow P$: If $x = 2$, then $2x + 4 = 8$.

$Q \Leftarrow P$ is true. Note that $P \Rightarrow Q$ is also true.

Since $P \Rightarrow Q$ is true and $Q \Leftarrow P$ is also true, we say P and Q are equivalent ie $P \Leftrightarrow Q$.

(b) $P \Rightarrow Q$: If $\frac{x}{2} + 3 < 7\frac{1}{2}$, then $x < 9$. $Q \Leftarrow P$: If $x < 9$, then $\frac{x}{2} + 3 < 7\frac{1}{2}$

$Q \Leftarrow P$ is true. Also, $P \Rightarrow Q$ is true. Hence, P and Q are equivalent ie $P \Leftrightarrow Q$

Reflection

Explain how you would guide the SHS3 students to answer the following questions.

1. Form compound statements from each of the following pairs of simple statements using '*and*', and 'or'.

Determine whether the compound statements $P \wedge Q$ and $P \vee Q$ are true or false.

a. *P*: $2(x + 4) = 2x + 8$ Q: $(a^2)^3 = a^8$

b. *P*: 169 is a square number. Q"1728 is perfect cube

c. *P*: $a^2 - b^2 = (a - b)(a + b)$. Q: $(a + b)^2 = a^2 + b^2$

d. *P*: $18 \in \{\text{multiples of } 4\}$. Q: $12 \in \{\text{odd numbers}\}$.

2. Construct the implication and its converse for each pair of statements. Determine whether $P \Rightarrow Q$ and $P \Leftarrow Q$ are true. Use Venn diagram to illustrate where necessary.

a. P: x is a prime number. Q: x is divisible by 4.

b. P : α is an angle between 270° and 360°. Q: α is in the 4^{th} quadrant.

SESSION 6: THE CHAIN RULE AND VALIDITY OF IMPLICATIONS

We learnt in Sessions 4 and 5 about statements and their negations, and compound statements including implications and their converses. This session introduces you to the chain rule and how to determine the validity of implications.

Learning outcome(s)

By the end of the session, the participant will be able to construct the chain rule of implications and determine the validity or otherwise of implications or conclusions.

Chain Rule

We know that if 2 < 3 and 3 < 5 then we can conclude that 2 < 5. Also, if $A \subset B$ and $B \subset C$ then $A \subset C$. These are facts about *transitive laws*: $a < b$ and $b < c$ implies $a < c$; and $a = b$ and $b = c$ implies $a = c$. We will make use of this idea of transitive law to construct the chain rule of implication.

The chain rule states that if $P \Rightarrow Q$ and $Q \Rightarrow R$ then $P \Rightarrow R$. Symbolically, we write: $P \Rightarrow Q \Rightarrow R$.

Let P, Q, R be statements defined by P: $x < 3$ Q: $x < 5$ R: $x < 8$

Then, $P \Rightarrow Q$: If $x < 3$, then $x < 5$ and $Q \Rightarrow R$: If $x < 5$ then $x < 8$.

Guide students to use the Venn diagram to illustrate these implications.

U = {Integers}, A = {Numbers less than 3}, B = {Numbers less than 5} and C = {Numbers less than 8}

The diagram shows that $A \subset B$ and $B \subset C$. We form the chain $A \subset B \subset C$.

The conclusion is $A \subset C$. That is, numbers less than 3 are also less than 8.

The resulting chain rule is $P \Rightarrow Q$ and $Q \Rightarrow R$ implies $P \Rightarrow R$.

Hence, $P \Rightarrow Q \Rightarrow R$.

We can extend the chain rule to more than three statements.

The chain rule is often used in developing mathematical proofs.

Ask students to try the following question.

Question 11

Construct a chain rule from each of the following pairs of true implications.

a. If a student studies mathematics, then he/she is in the mathematics club.
If a student is in the mathematics club, he/she does not play tennis.

b. If x is an odd number, then x^2 is odd.
   If x^2 is odd, then $x+1$ is even

Discuss the solution with the class.

Solution

(a) We can form the simple statements.

P: A student studies mathematics.

Q: A student is in the mathematics club.

R: A student does not play tennis.

We thus have the implication in symbols as $P \Rightarrow Q$ and $Q \Rightarrow R$.

The chain rule is $P \Rightarrow Q \Rightarrow R$ and the conclusion is $P \Rightarrow R$.

That is, $P \Rightarrow R$: If a student studies mathematics, then he does not play tennis.

b. We form the simple statements:
   P: x is an odd number. Q: x^2 is odd. R: $x+1$ is even.

The implications in symbols are $P \Rightarrow Q$ and $Q \Rightarrow R$

We therefore form the chain rule $P \Rightarrow Q \Rightarrow R$ and conclude that $P \Rightarrow R$.

That is, $P \Rightarrow R$: If x is an odd number, then $x + 1$ is even.

Validity of implications

In proving a theorem in mathematics, we make a sequence of statements usually starting from some chosen statements and ending with a final statement called *conclusion.* In real life, we often argue cases out by making a series of related statements and establish a conclusion at the end. Often a premise is chosen and the argument is based on the given premise. We often hear people advising their children using the following statements:

1. Disobedient children find themselves in jail, so if Aleb is disobedient he will be jailed.
2. All brilliant students study hard, so if you study hard you will be brilliant.

Discuss the two statements with students. Guide them to obtain two statements from the first advice; *Disobedient children find themselves in jail, so if Abu is disobedient he will be jailed.*

P: Disobedient children find themselves in jail. Q: If Aleb is disobedient, he will be jailed.

Let them recognize statement P as the *major premise* – a form of generalization.

The compound statement Q is made up of two simple statements: Aleb is disobedient, and Aleb will be jailed.

The statement Aleb is disobedient' is a *specific case* and Aleb will be jailed is a form of *conclusion* based on the given premise and the specific case. We however have to determine whether the conclusion drawn is valid.

Explain to students that it is very useful to use Venn diagram to illustrate the statements and see clearly the validity or otherwise of the argument.

For example, we form the sets:

U = {Children} D = {Disobedient children} J = {Jailed children} *a* = Aleb

Guide students to notice from the diagram that $D \subset J$ and clearly there is only one position for Aleb, that is, in the set D; $a \in D$. Once Aleb is in set D he will be jailed. Hence, the argument or conclusion is *valid*.

Explain to students that from the second advice, we obtain the following statements.

P: All brilliant students study hard (the premise). Q: If you study hard, you will be brilliant.

Let them identify the two simple statements from Statement Q.

1. You study hard (specific case or observation)
2. You will be brilliant (a conclusion)

Guide them to represent this on the Venn diagram, using the sets

U = {Students}, B = {Brilliant Students},

H = {Students who study hard}, y = a particular student .

Guide students to observe the Venn diagram and explain the conclusion they draw.

That is, from the diagram $B \subset H$ but there are two possible positions for the particular student *y*. If you are the particular student who studies hard you can be outside set B but in H, the set of students who study hard. Hence, the conclusion or argument that if you study hard you will be brilliant is not valid. The conclusion does not hold always following from the premise. *If in a sequence of statements, each statement follows logically from the preceding ones we say the argument or conclusion is valid. An argument refers to the sequence of statements.*

Put students into small groups and ask them to answer the following question.

Question 12:

a) Form a valid conclusion form the following statements.

P: People from this village are Christians Q: Jojo is not a Christian.

b) State whether statement S is a valid conclusion from statement P and Q. Illustrate with a Venn diagram

P: All girls like studying English

Q: Those who study English speak it fluently

S: Hawah speaks English fluently since she is a girl.

Discuss the solution with the class.

Solution

a) We use Venn diagram to illustrate.

Let U = {People}, V = {People from this village},

C = {Christians} and j = Jojo

The diagram shows that $V \subset C$ and $j \notin C; \quad j \notin V$

We therefore conclude that Jojo is not from this village.

b) Let U = {People}, G = {girls}

E = {Those who study English}

F = {Those who speak English fluently}

h = Hawah

From the diagram $G \subset E \subset F$ and $h \in G$. Since $h \in G$, the statement S is a valid conclusion from P and Q.

Reflection

Explain how you guide the SHS3 students to answer the following questions.

1. Determine the validity of each of the following arguments. Use Venn diagram to illustrate

a. Rich farmers have cocoa farms. If a person has cocoa farm, he does not cultivate maize.

b. Humble ladies get good husbands. Betty has a good husband so she is a humble lady.

c. Every man in a certain village has more than three children. Abbey has two children so he is not a man from the village.

d. Lemons taste bitter. Guava has a bitter taste, so guava is a lemon.

2. Determine whether statement R is a valid conclusion from statements P and Q. Use Venn diagram to illustrate.

a. P: Any one who owns a house has a car. Q: No one with a house has a child.

R_1: Seini has a car so he does not have a child. R_2: Sablah has a child since he has no house.

b. P: Ghanaians who travel abroad visit the Netherlands.

Q: Mr. Kum, a Ghanaian travelled abroad. R: Mr. Kum visited the Netherlands.

c. P: Children from poor home do not attend a university

Q: Placca attended a university. R: Placca is not a child from a poor home.

3. Given the statements: *P:* $x > 5$, *Q:* $x > 4$ and *R:* $x > 3$, form a *chain P* ® *Q* ® *R* and state if
it is true.

[Answer *P* ® *Q:* $x > 5$ ® $x > 4$ (**true**).
Q ® *R:* $x > 4$ ® $x > 3$ (**true**).
The *chain* between these implications is *P* ® *Q* ® *R:* $x > 5$ ® $x > 4$ ® $x > 3$ (**true**).
From the chain we can deduce ***P*** ® ***R:*** $x > 5$ ® $x > 3$ (**true**).]