Module for Bachelor of Education Programme (Primary and JHS)

EBS289SW: NATURE OF MATHEMATICS

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UNIT 1: DEFINITION OF MATHEMATICS

Learners, consumers, and practitioners are introduced to the nature of mathematics as a reality, an abstract discipline, a creation or discovery of the human mind, and an art or science in this lesson. It talks about certain mathematical definitions. The lesson provides insight into the problem-solving approaches of George Polya and John Mason as well as what "doing mathematics" actually entails. You will learn about the nature of mathematics as a fact, as an abstract discipline, as a creation or discovery of the human mind, or as a science in this unit. It talks about certain mathematical definitions. The lesson also reveals the true nature of "doing mathematics" and the approach taken by George Polya and John Mason to solve problems.

Learning objective(s)

The participant will be able to:

1. explain the definitions of mathematics;
2. explain the key similarities and differences between mathematics and science.
3. specify the elements of the cycle of mathematical investigations and explain them.
4. justify the notion that mathematics is a discovery or an invention.

SESSION 1: WHAT IS MATHEMATICS?

We will explain the definitions of mathematics in this session, as well as the difference between abstract and real mathematics. By the end of the session, it is hoped that you will have a better knowledge of what mathematics is all about.

Learning outcomes

By the end of the session, the participant will be able to explain:

1. definitions of mathematics; and
2. if mathematics is real or abstract.

WHAT DOES MATHEMATICS MEAN?

Of all the academic disciplines, mathematics is one of the oldest. It is frequently mentioned, used, admired, and criticised, and it has long been regarded as one of the most important aspects of human thought. The definition of the word "mathematics" has changed significantly over time and among different people. The definition of mathematics is not generally accepted. Depending on the type of research the definer has done, different definitions of mathematics exist. Each definition identifies the mathematical concept that the researcher favours. This suggests that our conceptions of mathematics largely depend on our individual knowledge and experiences in the field. Some people may only be able to perform addition, subtraction, multiplication, and division calculations. Some people might desire to incorporate trigonometry, algebra, and geometry. Others believe it calls for logical reasoning. From all of these, it is clear that mathematics is employed to find solutions to issues and questions that come up in daily life as well as in various industries and professions. In this lesson, we'll talk about a few mathematical definitions.

Mathematics is the logical study of shape, arrangement, amount, and many related ideas, per the James and James Dictionary of Mathematics. The three subfields are algebra, analysis, and geometry. Branch divisions are impossible since they are completely entangled. Geometry deals with space and related ideas, and analysis with continuity and limits, while algebra deals with numbers and their abstractions. Technically speaking, mathematics is a postulational science where necessary inferences are made based on predetermined premises.

Mathematics is viewed as a science of amount and space, where these concepts are expressed through symbolic representations. It is a science that involves drawing broad generalisations regarding quantity and area. The term "quantity" refers to calculations and arithmetic. Geometry, spatial correlations, and theories, such as the science of measurement and deductive science using axioms, definitions, and arguments, are all covered by the concept of space. Observations of patterns, presumptions, deductions, and conclusions follow from general findings.

While being of enormous significance in of of itself, mathematics may be considered as the benefactor of other fields. As a result, mathematics has a global shape. Therefore, the people who use mathematics determine its nature. Because it offers suggestions for scientific extension, it is known as the "queen" of the sciences.

According to Morris Kline, mathematics is a creative or imaginative process that derives concepts and recommendations from actual issues. The method is built on intuition and construction, with real-world situations serving as its life source. The abstractions will be taken from the actual issues and will therefore have a clear purpose in the context. According to Kline, the origin of the abstract idea can be found in the physical world.

It is obvious that the material cosmos itself is the greatest of our mathematical creations, and that mathematics is truly physical in nature. A symbolic depiction of physical reality is mathematics. It need brains and learning capacity to succeed in mathematics. The ability to successfully express physical reality with symbols is a creation or discovery of intelligence. New mathematical discoveries only result in an improvement in the way symbols are used to describe reality.

The most powerful of all theoretical systems, according to Richard Skemp, is mathematics since it is the most abstract. Therefore, it has the greatest potential for usage. Engineers, scientists, economists, navigators, businesspeople, and especially scientists regard it as a vital "tool" (data processing tool) for their work. The primary issue with mathematics is the degree of abstraction and generality that has been reached by consecutive generations of exceptionally clever people, each of whom has been generalising or abstracting from ideas of preceding generations.

Bertrand Russell, a distinguished English mathematician and philosopher, described mathematics as "*the topic in which we never know what we are talking about nor whether what we are saying is true*." Typically, we start explanations or discoveries with undefined terms (such as "point," "line," etc.), try to explain other things in terms of these undefined terms, and then make propositions.

Logic and creativity are key components of mathematics, which is studied for both its intrinsic appeal and a wide range of practical applications. The appeal and intellectual challenge of mathematics are for some people the essence of the subject. Others see mathematics' main importance as how it relates to their own work. Scientific literacy requires at least a basic comprehension of mathematics' nature because it is so important to contemporary culture. To do this, students must understand that mathematics is a branch of science, understand the nature of mathematical thought, and grow used to fundamental mathematical concepts and techniques It is clear that mathematics has a wide range of applications. A science with numerous facets is mathematics. It is a magnificent feat of human thought. It is impossible to define it in one sentence or a few sentences. But by looking at it from numerous angles and doing some of the things that insiders do, the outsider might gradually build a rich understanding of the nature of mathematics. It exhorts educators to get their students involved in mathematical activities.

Our understanding of the world is organised through the use of mathematics. It deepens our comprehension, improves our ability to communicate, and helps us make sense of our experiences. We also find enjoyment in it. We can accomplish a variety of practical tasks and solve real-world issues by using mathematics. It is utilised widely in our daily lives. In mathematics, we speak both regular English and a unique mathematical language. Students must be taught to use both languages. Problems in science, economics, geography, and other fields that employ mathematics as a tool are examples of problems that we can work on. In addition to describing and explaining, mathematics can also make future predictions. That is why mathematics is significant.

Key ideas

Key I

- Mathematics is a word whose meaning has varied widely from time to time and from person to person.
- Mathematics is the logical study of shape, arrangement, quantity and many related concepts". It is divided into three fields: algebra, analysis, and geometry
- Mathematics is regarded as a science of quantity and space where symbolic forms are used to express them.
- Mathematics relies on logic and creativity, and it is pursued both for a variety of practical purposes and for its intrinsic interest
- Mathematics is a way of organising our experiences of the world. It enriches our understanding and enables us to communicate and make sense of our experiences

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to achieve the school curricula aims, values and aspirations? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How similar are the aims of the subjects discussed in the mathematics curriculum? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum?

SESSION 2: SCIENCE, TECHNOLOGY AND MATHEMATICS

We will explain science, technology, and math in this session, as well as how they relate to one another. It is desired that students will be able to describe how science, technology, and mathematics are related in the context of their instruction and how it applies to the outside world.

Learning outcomes

The participant will be able to explain the:

1. distinction between science and mathematics; science and technology; and technology and
   mathematics; and
2. relationship between science, technology, and mathematics

Natural science exclusively considers patterns that are applicable to the observable world, whereas mathematics studies all patterns or correlations. Despite having its roots in practical issues, mathematics rapidly turned to abstractions from the real world and subsequently even more abstract interactions between those abstractions.

The belief in underlying order, the standards of honesty and transparency in research reporting, the usefulness of peer review in determining the worth of new work, and the crucial role of imagination are only a few of the characteristics that mathematics has with other sciences. Mathematics, like science, involves both figuring out the answers to fundamental questions and resolving real-world issues. People can think about the universe of things and events using mathematics, and they can express their thoughts in ways that show unity and order. The realm of mathematics, which consists of numbers, lines, angles, shapes, dimensions, averages, probabilities, ratios, operations, correlations, etc., helps individuals make sense of a cosmos that would otherwise appear to be utterly complex. Over the ages, mathematical correlations and patterns have been created and improved, and this work is still being done actively now. More than ever before, mathematics is used in a variety of subjects of study and has also taken on greater importance in daily life.

Practically speaking, mathematics is the study of order and pattern. Numbers, chance, form, algorithms, and change all fall under its purview. Although mathematics uses simulation, experimentation, and even observation as a way to find the truth, it uses logic as its criterion of truth rather than observation (Mathematical Sciences Education Board, 1989 p. 31). Due to its global applicability, mathematics has a unique place in education. The outcomes of mathematics are important and helpful. For instance, mathematics provides science with theorems that serve as both a criterion of certainty and a basis for truth. As a result, the language of mathematics is a magnificent gift in the creation of the rules of physics. Every aspect of modern science bears the indelible mark of mathematics. Cross-fertilization between science and mathematics occurs in problems, theories, and notions whether it is intentional or not. This is at its greatest point ever.

Therefore, all pupils should have the opportunity to learn for themselves how an idea might be expressed in various but comparable ways. Making several representations of the same concept and translating them from one to another are important learning strategies, according to one branch of study on how individuals learn. One may be sure that a pupil has truly understood a relationship when they can start to represent it in tables, graphs, symbols, and phrases. Students can practise making those representations and translations by seeing them in situations when the solution matters to them. Through this kind of work, students will finally understand how mathematics is connected. The primary goal of mathematics instruction ought to be this. Again, simplicity is one of the highest qualities in mathematics, just as it is in the sciences. The minimal set of principles from which numerous other statements can be rationally deduced is the focus of some mathematicians. New mathematical theories can occasionally be generated in response to real-world issues, but they also frequently have applicability in real world situations.

Without considering its utility, mathematics is frequently practised for its own sake. However, the majority of mathematics does have applications, and practical issues frequently inspire new mathematical research. A significant portion of these applications and stimulants are provided by science and technology. Scientists and engineers may try to perform some practical mathematics themselves while working or may seek the assistance of mathematicians. Help could come in the form of recommending some already-completed mathematics that will work or by creating some new mathematics that will. On the one hand, there have been some amazing instances of repurposing mathematical concepts from previous eras. On the other hand, new mathematics have frequently been developed in response to the demands of natural science or technology.

Science, technology, and mathematics should all work together in the classroom to assist students comprehend the value of math and science. Students will understand the value of mathematics in science and technology if they regularly engage with it in both basic and later complex forms. The contexts of science and technology are particularly rich and significant for teaching the value of mathematics and for honing mathematical problem-solving abilities. It is appropriate to learn and use mathematics in courses like music, social studies, history, physical education, sports, driver education, and home economics.

Key ideas

Key II
• Mathematics is the study of any patterns or relationships, whereas natural science is concerned only with those patterns that are relevant to the observable world. • Mathematics shares many of the features of other sciences, such as the belief in an underlying order, the ideals of honesty and openness in reporting research, the importance of criticism by colleagues in judging the value of new work, and the essential role played by imagination. • Science and technology are rich and especially important contexts in which students learn the value of mathematics and enable them to develop mathematical problem-solving skills. • Natural science or technology has often led to the formulation of new mathematics.

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to achieve the school curriculum aims, values, and aspirations of Science, Technology and mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How similar are the aims of the Science, technology and mathematics discussed in the mathematics curriculum? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 3: CYCLE OF MATHEMATICAL INVESTIGATION

The three parts of the cycle of mathematical investigation will also be discussed, together with an explanation of mathematical investigation, in this session. It is envisaged that after completing real-world problems, students will be able to explain the three parts of mathematical investigation.

Learning results

The participant will have the ability to clarify by the end of the session:

1. The mathematical investigation cycle

2. The three elements of the mathematical inquiry cycle (i.e., representation, manipulation, and validation)

Although it is crucial for students to learn how to answer specific types of well-defined mathematical problems, this does not inevitably result in a thorough knowledge of how mathematical investigations work. It is possible to define mathematics as an ongoing cycle of research that aims to produce sound mathematical concepts. Although certain procedures are included in mathematical research, the order is not defined, and the importance given to each phase varies substantially when addressing problems in the actual world. The cycle consists of three parts: validation, manipulation, and representation. The three components of the cycle should each be studied separately as a component of studying mathematics. The full cycle should be available to students so they can conduct their own mathematical explorations. This event aims to generate adults who are conversant with mathematical enquiry rather than professional mathematicians.

Many students interpret the definition of representation—which is the act of representing something through a symbol or expression—to mean only "actual things." Young pupils will have an easier time understanding "let A equal the area of any rectangle" than "let A stand for the area of the floor of a room." Students must first be persuaded that the work involved in replacing abstract symbols with precise quantities is worthwhile. Then, kids must gradually come to the understanding that utilising symbols to represent abstractions and abstractions of abstractions is useful for problem-solving as well. This could imply showing children that in the realm of mathematics, numbers, shapes, operations, symbols, and symbols that sum up sets of symbols are just as "real" as blocks, cattle, and cedis, dollars, and pounds.

Students manipulate symbols by moving them around in an organised way to find a solution to a problem. When manipulating, students should keep in mind that there are always regulations that must be followed to the letter and that these rules are subject to change. That is where mathematics' rigour and competitive spirit converge. Set up a problem, imagine some quantities, give them qualities, choose some operations, and represent everything with symbols. Then, using the logic principles that have been selected, rearrange the symbols to see what answers appear. Finding solutions to difficulties in daily life is aided by this technique.

Validation examines the quality of the solutions. Students are accustomed to solving mathematical puzzles with predetermined steps and "proper" solutions. A good solution, however, is one that leads to new mathematical discoveries or to useful outcomes in science or medicine in actual mathematical studies. engineering, commerce, or somewhere else. As a result, judgement, not authority, is required for validation in mathematics.

The cycle of study makes use of concrete materials to serve practical needs. Students should frequently be guided by concrete items to identify and explain symbolic linkages. The ability to use numbers and shapes to describe a variety of objects in their environment should dawn on students. They should eventually realise that just as words and letters in reading and writing make up a language, so do numbers and shapes in mathematics. To help students make the connection between tangible items and their abstract representations, concrete objects must still be used often in lessons. Frequent mention of real-world applications will improve their capacity to visualise and carry out tasks in their minds. Encourage your students to use numbers, shapes, and operations to explain anything and everything.

Key ideas

Key III
• Mathematics investigation is a process • True mathematical investigations involve certain processes, but the order is not fixed and the emphasis placed on each process varies greatly in solving real life problems. • **Representation** is a process of representing something by a symbol or expression and this is taken by many students to refer only to "real things. • **Manipulation** involves students in moving symbols about in a certain ordered manner to arrive at a solution to a problem • **Validation** deals with *how good the solutions are*. Students are used to working mathematical problems in which the procedures are predetermined and "correct" answers are expected.

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to do mathematics through mathematical investigations? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How are the three components of the cycle of mathematical investigation related? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum) by teaching mathematics through mathematical investigations?

SESSION 4: MATHEMATICS AS AN INVENTION AND AS A DISCOVERY

In this lesson, we'll concentrate on defining mathematics as either an invention or a discovery and discussing how the two relate to one another. In the context of mathematics education, it is intended that students would be able to describe mathematics as an invention and a discovery.
Learning results
The participant will be able to describe the following by the end of the session: the relationship between mathematics as an invention and as a discovery

The fundamental ideas in mathematics are abstractions of real-world experiences, such as the physical counterparts of whole numbers and fractions. Others were concocted by the human intellect, with or without some assistance from experience. For instance, mathematicians created the irrational number 2 to symbolise the hypotenuse of a right-angled triangle with both arms being one unit long. Other options include negative numbers and variables to reflect temperature changes and other shifting physical phenomena like ds/dt. The Babylonians and the Egyptians are credited with inventing or creating numbers and numerals. Their own numerals were created by the Mayans and Romans. The numbers differ from one group of people to another, but the structure that already exists and demonstrates the relationship between these numbers cannot be handled physically. It is necessary to find the structures and do this. leads to mathematics' discovery-based character, but proofs, operations, numerals, and other inventions of the human intellect are also inventions.

Areas, perimeters, and other concepts like these all exist in reality since it is something that can exist. The discovery of all these will heavily rely on the human mind. For instance, Leibnitz and others are credited with the discovery of Calculus, Newton with the discovery of mechanics, and Galois with the development of group theory. According to one school of thinking, mathematicians uncover the principles and laws of mathematics, which are found in nature just as certain physical laws are found there. The opposing school holds that mathematics is more akin to a piece of art, such as a painting, which doesn't exist until it is created by the artist, in this case a mathematician. Because mathematics requires as much creativity as art does, it is viewed as an art. Like a painter or poet, we can create beauty in mathematics by using patterns, but unlike these artists, the patterns used by mathematicians are made of ideas rather than words, which last longer over time (G. H. Hardy, A Mathematician's Apology). This is why mathematical beauty lasts longer than artistic beauty.

Key ideas

Key I

- Mathematics emerged out of an invention or a discovery
- The structures of mathematics need to be discovered and that leads to the discovery nature of mathematics.
- Human mind plays a great role in the discovery of mathematics.
- Mathematical proofs, operations, numerals etc are invented and so are creations of the human mind
- Mathematics is regarded as an art because we use a lot of imagination in mathematics as we do in art.
- We create beauty in mathematics, using patterns like a painter or a poet, but mathematical beauty is more lasting than that of art because unlike the poet or painter.

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to appreciate mathematics as an invention and discovery?
- How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?

Discussion

- How has this session equipped you to be a better classroom practitioner?
- How has your idea of mathematics as an invention and as a discovery equipped you to be a better mathematics teacher?
- How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum) in context of mathematics as an invention and as a discovery?

UNIT 2: SOME MATHEMATICS EDUCATION TERMINOLOGIES

We spoke about several definitions of mathematics and the basics of mathematics in unit 2. During the conversation, a few terms related to mathematical education were used. We will define a few of the terms used frequently in mathematics instruction in this unit. We will go over how some of the terminologies are similar and different.

Learning outcome(s)

By the end of the unit, you should be able to:

1. explain mathematical axioms and provide examples;
2. recognise and explain mathematical operations,
3. recognize and explain a mathematical proof,
4. recognise and explain mathematical algorithms,
5. differentiate between a mathematical conjecture and theorem, and
6. recognise and explain mathematical paradoxes/antinomies.

SESSION 1: MATHEMATICS AXIOMS

In this session, we'll concentrate on defining mathematical axioms and discussing how they apply to math instruction. It is envisaged that students will be able to explain mathematics as it is taught in math classes.

Learning outcomes

By the end of the session, the participant will be capable of

1. explaining mathematical axioms; and
2. use the idea of axioms in teaching and learning mathematics.

The Greek word "axios," which means "something worthy," is where the word "axiom" comes from. A statement that is taken as true is referred to as an axiom in mathematics. They are assertions made in relation to undefined words that are regarded as obvious truths. Axioms are assertions that appear to apply to a fundamental idea. They are widespread beliefs that must be acknowledged because of the way in which human cognition operates. The foundation of a mathematical theorem is an axiom. Simply said, axioms are viewed as universal truths.

Euclid was reputed to have taught mathematics at Alexandria University in 300 BC. He published a book titled "Elements." This book was viewed as an introduction manual that covered all of primary mathematics, including algebra, geometry, and arithmetic (not symbolic, but geometrical). Only the Bible is supposed to have had more printings than The Element, which is considered to be the most successful mathematics textbook in history. Since Euclid made various rational claims that were uncontested, his name came to be connected with truth; "Euclid is truth."

The following are included in Euclid's set of axioms known as the "Five Common Notions":

1. Things that are equivalent to one another are also equivalent to that object. For instance, if 3 + 2 = 5 and 4 + 1 = 5, then 3 + 2 = 4 + 1 as each equal 5.
2. The wholes are equal if equal things should be added to equal things. For instance, if $a = b, a + x = b + x$.
3. The remainders are equal when equals are subtracted from equals. If $a = b, a - x = b - x$.
4. Things that are in sync with one another are on an equal footing. Eg $3 + 2 = 2 + 3$. Things that fit together equally, in other words, are equal to one another. By "fitting one object to another," Euclid presumably implies imaginarily picking up, for example, a triangle and setting it down upon a comparable triangle to see if all the points match up.
5. Greater than the sum of its parts is the totality.

Noting that none of these seem to require any proof, they all seem to be self-explanatory.

SESSION 2: MATHEMATICAL OPERATIONS

The explanation of mathematical operations and their uses in math instruction will be the main topics of this session. It is envisaged that students would be able to employ mathematical operations in the way that they are taught.

Learning outcomes

The participant will be able to:

1. describe mathematical operations by the end of the session, and
2. use the idea of operations in the teaching and learning of mathematics

A process involving a change or transformation is referred to as an "operation." The process begins with an object in a specific state of affairs; an operation is performed, causing the object to change, which leads to a final state of affairs. Thus, an input-output scenario is created. Between the input time and the output stage, an operation takes place. Any process used on one or more initial values (the operands) to produce a new value is referred to as an operation.

Mathematics is based on the concept of an operation. The process of applying procedural rules, such as addition, subtraction, and multiplication, is known as a mathematical operation. and discord. Other operations include squaring, higher powers, differentiating, integrating, and using square, cube, and other types of roots. Each of these processes includes the manipulation of mathematical variables, numbers, or other things. Each has a distinct underlying structure to maintain, which guides what has to be done. When the addition operation is performed on the operands 3 and 4, the result is a sum of 7. Even quite algebraic approaches like factorization require a solid grasp of fundamental operations.

There is always a precise formula for working out the outcome of a specific operation. Regardless of the quantity of input values, the result of numerous operations is always one value. The square root procedure is an exception, as the results can be either positive or negative. Such operations can be categorised as functions, one-to-one or many-to-one mappings:

An operator is a symbol that represents a particular operation. The plus sign (+) serves as the operator for addition, while the integral sign serves as the operator for integration. Sometimes the same operators are represented by various symbols. In computing, the operator * stands for the same thing as the mathematical operator, "times," denotes. In many branches of mathematics, different operators are employed. For instance, logical relationships can be expressed using a variety of sets of operators.

Addition

Putting together or connecting two things is what the addition operation entails. You must combine the entities that are participating in the addition process, according to the procedure. As an illustration, to add 5 and 4, or 5 + 4, we first depict 5 as 3 concrete objects and 4 as 4 concrete items, then we combine them and add them all up to get 9 concrete pieces. The first addend (5) and the second addend (4) are counted individually, combined, and then all are counted to arrive at the result. This method is frequently known as the "counting all principle" (9). In other words, to create a collection of x items and a collection of y objects, combine the two collections, and then count all the things to create the $x + y$ objects.

At a higher level, the counting-on technique can be used to calculate 5 + 4. In this method, you start with one of the addends (typically the higher number), say 5, and then just count on four from that number using some objects or fingers or just mentally, stating 5: 6, 7, 8, 9, and then give the last count, nine, as the solution. The binary nature of addition means that it can only combine two quantities at once. As a result, when adding three or more numbers, we pair the numbers together until they are all added.

Subtraction

A defined quantity is subtracted from a bigger collection of elements in a subtraction operation. The Take Away Aspect of Subtraction is a common name for this. Taking away from or removing from is the proper course of action in this situation. For instance, $9 - 4 =?$ means that there are 9 things on the table. Kwame removes four of the objects. How many things are still on the table, flow?

Comparing and matching two sets is a second way to look at subtraction. This involves comparing the number of items in two collections or sets by matching them one to one, and then counting the number of extra items in the larger set. For instance, Kwesi has 5 similar items, whereas Ama has 9. Ama has how many more items than Kwesi? For 9-5?).

A third interpretation of subtraction operation is referred to as Missing Addend. This requires finding how many more items to be added to one collection to get the number of items in the second collection. E.g. Stephanie has 6 books and Robert has 9 books. How many more books does Stephanie need to get as many books as Robert? ($for\ 6+? = 9\ or\ 9 - 6?$).

In each of the three cases, the subtraction operation requires that we look for the difference in the two given numbers.

Multiplication

Multiplication can be understood as combining sets or making the same addition repeatedly.

In order to multiply as mixing sets, two sets must be created and each member of the first set must be matched with every member of the second set. This is also known as a Cartesian product, where all of the members of one set are matched with all of the members of another set in ordered pairs. For instance, Musa owns 6 shirts with various colours and 4 sets of pants with various colours. How many outfits can he wear for a day out? This entails combining each pair of trousers with each shirt, resulting in a total of $4 \times 6 = 24$ pairs or combinations.

The same number must be added again a certain number of times in order for multiplication to function as repeated addition. For instance, 64 is translated as $4 + 4 + 4 + 4 + 4 + 4 = 24$. This is frequently understood to mean "6 lots of 4." $6x4$ translates to 6 lots of $4 - 4 + 4 + 4 + 4 + 4 + 4 = 24$.

Additionally, 3×9 indicates that there are 3 lots of 9, which add up to 27. Students need consistency in the interpretation in order to establish the commutative law, such as $3 \times 9 = 9 \times 3$.

Division

Division can be seen as both a measurement problem and a partition problem.

We are dealing with the measurement component of division when both the total number of things and the number of items to be placed in each group are specified. The number of groups that can be formed from the bigger collection must be determined. This results in the grouping component of division, which gives rise to the idea of repeated subtraction. For instance, $15 \div 5 = 3$ can be understood as how many students would receive the mangoes if there are 15 mangoes and each student is to receive 5 mangoes.

The partition component of division requires the provision of both the total number of items and the number of groups to be formed. In this instance, we must determine how many items each group will include. This part of division is also known as the "sharing" aspect. 15 apples, for instance, are to be distributed equally among 5 students. Each pupil will receive how many oranges? This clarifies the idea of "take one, I take one" sharing.

"One of the two equal factors of the given integer," is how the square root of a number is defined. We attempt to factorise a given integer so that there are two equal factors using this interpretation of the square root. We must identify the equal elements of the supplied integer in order to do the square root function. For instance, since 81=99, we say that the square root of 81 is 9 and that the square of 9 is 81. By dividing the factors into two equal groups after prime factorising the provided number, this procedure can be carried out methodically. The square root of the product of each group's factors is then determined. To find 576, for instance, the technique begins by prime factorising 576 as

$576 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3$

$576 = (2223)(2223) = 2424.$
$Hence\ \sqrt{(576)} = 24.$

By dividing the prime factors into three equal groups, we may expand this to include the cube root of a particular integer as one of its three equal components. Try the formula $\sqrt[3]{216} =$? $216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3\ =\ (2 \times 3) \times (2 \times 3) \times (2 \times 3). Hence, \sqrt[3]{216} = 2 \times 3 =$ 6, one of the three equal factors.

Key ideas

Key I • Mathematical operation is the process of carrying out rules of procedure such as addition, subtraction, multiplication. And division. Others are squaring, and higher powers; roots such as square roots, cube roots, etc, differentiating, integrating, etc. • The symbol used to indicate an operation is called an operator • Addition operation means putting together, or joining two entities • The subtraction operation is the process of removing a specified quantity from a given larger collection of items • Multiplication as repeated addition requires that the same number be added repeatedly for a required number of times • Division can be interpreted as a measurement problem and as a partition problem

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply mathematical operations in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of mathematical operations equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 3: MATHEMATICAL ALGORITHMS

In this session, we'll concentrate on defining mathematical algorithms and discussing how they might be used in math instruction. It is envisaged that students would be able to utilise the mathematical algorithms taught in math classes.

Learning outcomes

The attendee will be able to:

1. describe mathematical algorithms by the end of the session.
2. apply the idea of mathematical algorithms to math instruction and learning.

A mathematical algorithm is a finite set of specific instructions that can be used to make calculations or to solve problems, including all possible solutions. It usually describes a series of actions taken to carry out a mathematical calculation. It is a step-by-step process with as many repetitions as necessary that aims to accomplish a specific goal in a set amount of time. The algorithms for addition, subtraction, multiplication, and division are the most often used. Students can use the following procedures, for example, to add the single-digit numbers $4 + 7 + 6$: $4 + 7 + 6 = 11 + 6 = 17$ (starting from the left); $4 + 7 + 6 = 4 + 13 = 17$ (starting from the right); $4 + 7 + 6 - 7 + 10 = 17$; (Looking for 10).

HMMDIA

"How Much More Do I Add" is the name of one (not very common) subtraction method (HMMDIA). The algorithm uses an additive component, increasing the quantity to be subtracted while subtracting it until you reach the minimum (number from which we are subtracting). For instance, the question "136 - 27=?" might be read as "How Much Do I Add to 37 to Get 146." This is calculated by adding each number starting at 37 up to 146 as follows:

First, how much more do I need to add to 37 to get the number 40 (the nearest ten)?

Note the response 3.

In order to obtain 130, how much more do I need to add to 60?

Note the response, 70.

In order to obtain 146, how much more do I need to add to 110?

Keep the response 36.

Add the outcomes now. 3+ 70+ 36 = 109

This means that 146 - 37 Equals 109.

Based on structured materials like the abacus and Dienes' blocks, we have algorithms for multiplication that demonstrate the methodical technique to employ when multiplying one-digit, two-digit, three-digit, etc. factors, with or without regrouping and using extended forms. There are also numerous algorithms for scaffolding and regrouping when dividing numbers by one-digit, two-digit, etc. divisors.

In lengthy division, a step-by-step process is employed. The algorithm for a straightforward calculation like 95 divided by 4 (95÷4) is as follows:

4 can be divided (goes into) 9 how many times?

How many are still remaining after the response of 2? 1

the 1 (ten) should come before the 5.

How many times is 15 divided by 4?

The answer is 3, leaving a 3 as the remainder.

The result is 23 with a remainder of 3, thus the equation is $95 \div 4 = 23$ remainder 3. This step-by-step method is referred to as a long-division algorithm.

FOIL

The acronym "FOIL," which stands for "First Outside, Inside Last," is another helpful example of an algebraic algorithm for multiplying polynomials. It is a useful method for keeping in mind how to multiply two binomials. For instance, the method requires that we multiply the First terms (a and c), or ac, before multiplying the Outside terms (a and d), or ad.

multiplying the last terms (b and d), bd, after multiplying the inside terms $(b\ and\ c), bc$. Finally, we calculate the total of all the outcomes (products). $(axc) + (axd) + (bxc) + (bxd)$. Thus, $(a + b)(c + d) = ac + ad + be + bd$ is produced. Specifically, we use the FOIL algorithm to find the product $(2x + 3)(5x - 6)$ as follows:

Multiply the First terms by $2x(5x)$ to get $10x^2$, the Outside terms by $2x(-6)$ to get $-12x$, the Inside terms by $3 \times (5x)$ to get $15x$, and the Last terms by $3 \times (-6)$ to get $18x$.

Find $10x^2 - 12x + 15x - 18$, which is the sum of all the results. This can be expressed as $(2x + 3)(5x - 6) = 10x^2 + 3x - 18$.

BEDMAS/PEDMAS

Mathematicians have agreed that the BEDMAS/PEDMAS sequence is appropriate when solving mathematical problems that call for the use of various operations (brackets or parenthesis, exponents, division, multiplication, addition, and subtraction, among others). The BEDMAS alphabet uses letters to represent various components of the operation. In mathematics, the sequence in which your operations are carried out is governed by a set of rules. If you make calculations out of order, your solution is probably incorrect. When using the BEDMAS order of operations, keep in mind that you should move from left to right. Exponents are always listed after brackets or parentheses. Working from left to right, you multiply or divide according to whatever comes first. If multiplication comes before division, perform it first. The same is true for addition and subtraction; subtract before adding when the subtraction comes first.

When there are many pairs of parentheses, start with the inner pair and work your way to the outer pair. (a) As an illustration, $26 + [8(7 - 3)]$

First, complete the inside bracket (parenthesis): $26 + [8 \times 4]$.

Complete the last bracket: $26 + 32$.

Add the number: 58.

Consequently, 26 + [8(7-3)] = 58

(b) $11 - (6 + 7)^2 + 6 \times 29$
Put the parenthesis in place as follows: $11 - (13)^2 + 6 \times 29$.

Exponent calculation for $11 - 169 + 6 \times 29$.

Multiply $11 - 169 \; + \; 174$ now.

Subtracting now: $-158 + 174$

Add: 16 now

Consequently, $11 - (6 + 7)^2 + 6 \times 29 = 16$.

The acronym BEDMAS, which stands for Brackets of Division, Multiplication, Addition, and Subtraction, is comparable to the well-known BODMAS (or BOMDAS).

Algorithms' role in school mathematics is shifting due to the accessibility of calculators and computers outside of the classroom. Learning a single standard algorithm for every operation, especially early on, may actually impede students' ability to gain a deeper knowledge of mathematics. We must introduce pupils to various algorithms and motivate them to create their own. This would encourage students to avoid using algorithms to replace critical thinking and common sense. Students must be helped to approach math issues in multiple ways. This gives children versatility in mathematics and aids in their development of computational skills. We must reach every student; while one algorithm might be effective for one student, another algorithm might be more effective for that student. In order to effectively teach and understand mathematics, multiple algorithm approaches must be used. Studying many algorithms for an operation will assist students comprehend the operation because various algorithms are frequently based on various notions.

Furthermore, offering a variety of possible methods conveys the idea that mathematics is both logical and inventive. Giving pupils multiple algorithms for key operations prepares them to practise mathematics outside of the classroom. A goal of school mathematics is to develop pupils' capacity for maths by teaching them algorithms (NCTM, 1989). When students create an efficient algorithm, they can utilise it to quickly tackle a variety of connected issues without having to approach each one from scratch.

Key ideas

Key I

- Mathematical algorithm is the process of carrying out rules of procedure such as HMMDIA, FOIL and BEDMAS/PEDMA
- Teaching several algorithms for important operations equips students to do mathematics outside the classroom

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply mathematical algorithms in the teaching and learning of mathematics?

• How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of mathematical operations equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 4: MATHEMATICAL CONJECTURE

In this session, we'll concentrate on defining mathematical conjecture and discussing how it can be used in math instruction. It is envisaged that students would be able to use mathematical conjecture in their coursework.

Learning outcomes

The participant will be able to explain mathematical conjecture and apply the idea to the teaching and learning of mathematics by the end of the session.

A statement with an undetermined truth value is referred to as a conjecture/hypothesis.
It is a claim that has not yet been established as true or false. It is an unsupported assertion that is thought to be true or that seems to be accurate. It is used to characterise situations where an inference or judgement is made based on speculation, shaky evidence, or both. A speculation is an inference drawn from circumstantial evidence, a guess, a conclusion drawn from insufficient data, or an assertion made without supporting evidence. A simple statement that someone believes to be supported by evidence is referred to be a hypothesis. The key characteristic of a supposition is the absence of any supporting evidence.

A prime number only has itself and one other factor. Composite numbers are those that include additional components. Here are a few hypothetical scenarios involving prime numbers to illustrate how educated predictions can be made and afterwards improved upon.

1. The conjecture of the quadratic function $f(n) = n^2 - n + 41$ for generating primes. This was later proved to work for the first 40 natural numbers (n) but not beyond though this has been accepted for quite a long time.

2. **Goldbach's even number conjecture:** Any even number greater than 2 is the sum of two primes e.g. $4 = 2 + 2;\ 6 = 3 + 3;\ 8 = 3 + 5;\ 10 = 3 + 7; 28 = 11 + 17;\ 48 = 5 + 43;\ 78 = 7 + 71 = 17 + 61;\ 102 = 5 + 97 = 19 + 83$; etc.

3. **Fermat's numbers** in generating prime numbers. In 1640, Fermat, a French mathematician, who was lawyer by profession but an amateur mathematician in his spare time, conjectured that 21 is a form which represented primes only. Check to see that the first four,
   $2^2 + 1 = 5;$
   $2^{2^2} + 1 = 17;$
   $2^{2^3} + 1 = 257,$ and
   $2^{2^4} = 65537$ are prime numbers

Hundred years later, Euler Leonhard (1707-1783) disproved this by showing that the fifth Fermat's number, $2^{2^5} + 1 = 4{,}294{,}967{,}296 \ + 1$, is not a prime. It is divisible by 641.

- In 1970, a Russian named Matsyasievich discovered several explicit polynomials of this sort that generate only primes. The largest was $2^{11213} - 1$, discovered at Illinois University.
- In 1971 (March 4), Bryant Tuckerman found another prime to be $2^{19937} - 1$.
- In 1978 two students, Laura Nickel and Curt Noll found a larger prime $2^{21701} - 1$. Later Curt found

$2^{23209} - 1$ as a prime.

- Then in 1983, David Slowinski (of the Cray Research Laboratory) found $2^{86243} - 1$ to be the largest known prime. This was a 3000-year-old puzzle solved (Los Angeles Times).

The largest prime known has been an integer the special form $2^p - 1$, where p is also a prime. Such primes are called **Mersenne primes**, after a French monk, Martin Mersenne who studied them in the 17^{th} century. After the discovery of the first few Mersenne Primes it took more than two centuries with rigorous verification to obtain 47 Mersenne primes. By mid - 1999, the largest Mersenne prime was $2^{3021377} - 1$.

4. **Fermat's 'two square' theorem**: - the Primes may (if we ignore the special prime 2) be arranged in two classes, the primes 5, 13, 17, 29, 37, 41, ...which leave a remainder 1 when divided by 4, and the primes 3, 7, 11, 19, 23, 31,..,which leave a remainder 3. All the primes of the first class and none of the second, can be expressed as the sum of two integral squares thus, $5 = 1^2 + 2^2$; $13 = 2^2 + 3^2$; $17 = 1^2 + 4^2$; $29 = 2^2 + 5^2$.
   This theorem is ranked one of the finest of arithmetic. Verify for some more numbers in the category.

5. The famous **Fermat's Last theorem** (Pierre de Fermat (1601-1665)

"When n is an integer bigger than 2, it is impossible to solve the equation $x^n + y^n = where\ x, y, z$ are integers".

Euler, Lagrange, Kummer and Riemann tried in vain to disprove or prove it. Eduard Kummer (1939) developed his theory of Ideal Numbers as a result. This has been a rumour for many centuries. Fermat asserted that he had the evidence. He is thought to have begun putting the suggestions in a margin but claimed he ran out of room before passing away. A 19-year-old British academic named Andrew from the Massachusetts Institute of Technology in Cambridge, United States, also asserted to have the evidence, but he insisted on 200 pages of foolscap paper. The "Twin Prime Conjecture," which asserts that "there exist an unlimited number of primes p such that p+2 is likewise a prime," and the "Odd Perfect Numbers Conjecture," which claims that "there are no odd perfect numbers," are a couple of examples. The countless primes of Euclid: - The Greek geometer Euclid hypothesised that there existed an endless number of primes when he was alive, approximately 300 B.C.

- Euler, Lagrange, Kummer and Riemann tried in vain to disprove or prove it. In so doing, Eduard Kummer (1939) created his theory of Ideal Numbers. For hundreds of years this remains a conjecture. Fermat himself claimed he had the proof. He is believed to have started writing the hints in a margin but claimed he hadn't enough space to complete it when he died.
- A 19 year old British professor Andrew (Massachusset Institute of Technology, Cambridge, USA) also claimed he had the proof but he needed 200 pages of foolscap paper to prove it.

Here is an example of its subsequent proof. Assume that the number of primes is finite. If this is the case, then P must be the greatest prime. Create the sum of all these primes, then

$$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times \ldots \times\ P + 1 = Q$$

Q will always have a leftover of 1, no matter which prime number we divide it by.

Since Q cannot be divided by any of these primes, it must either be a prime itself or a composite number that can be divided by a greater prime. (Q can be factored into primes if it is composite. Since none of these numbers are factors of Q, these primes could not be any of the numbers 2,3,5...,P. In either scenario, the initial assertion that P is the greatest prime is false, proving that there is another prime that is greater than P.

There can't be a finite number of primes. The product of all previous primes plus 1 generates another prime. Systematically, we have

$$2 \times 3 \times 5 + 1 = 31$$
$$2 \times 3 \times 5 \times 7 + 1 = 211$$
$$2 \times 3 \times 5 \times 7 \times 11 + 1 =$$
$$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times \ldots \times n + 1 = etc$$

When one number is divided by another that is a prime, there must be a remainder.

Key ideas

Key I
• Mathematical conjecture is a mathematical proposition that is yet to be proven or disproved. • A prime number only has itself and one other factor. • Composite numbers are those that include additional components

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply mathematical conjecture in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of mathematical conjectures equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 5: MATHEMATICAL THEOREM

In this session, we'll concentrate on defining mathematical theorems and discussing how they might be used in math instruction. It is envisaged that students would be able to apply theorems taught in mathematics classes.

Learning outcomes

The participant will be able to: clarify mathematical theorems and apply them to the teaching and learning of mathematics by the end of the session.

Mathematical Theorem

A proposition that has been shown to be true is called a theorem. A widely acknowledged rule or principle in mathematics is known as a theorem. A conjecture turns into a theorem once it has been demonstrated. Theorems are conclusions reached by deducing from premises and applying the axioms and canon laws of logic.

Here are three instances.

Theorem of Prime Numbers

The asymptotic distribution of the prime numbers is described by the prime number theorem (PNT). The theorem provides a general account of the distribution of prime numbers among positive integers. It formalises the intuitive notion that as primes get bigger, they become less common.

According to the theorem, the probability that a random integer chosen between zero and a large integer N would be a prime number is around $\frac{1}{lnN}$,, where ln(N) is the natural logarithm of N. As a result, the probability that a random integer with at most 2n digits is prime is roughly half that of a random integer with at most digits. For instance, among positive integers with a maximum of 1000 digits, roughly one in 2300 ($In\ 10^{1000} \approx 2302.6$) is a prime number, whereas among positive integers with a maximum of 2000 digits, about one in 4600 ($In\ 10^{2000} \approx 4605.2$). is. In other words, for the first N integers, the average distance between consecutive prime numbers is nearly equal to ln (N).

Binomial Theorem

The theorem states that any power of $x + y$ can be expanded into a sum of the form

$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^n y^n,$

where each $\binom{n}{k}$ is a specific positive integer known as a binomial coefficient. This formula is also referred to as the *binomial formula or the binomial identity.*

Using summation notation, it can be written as $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$

The final expression follows from the previous one by the symmetry of x and y in the first expression, and by comparison it follows that the sequence of binomial coefficients in the formula is symmetrical.

A simple variant of the binomial formula is obtained by substituting 1 for y, so that it involves only a single variable. In this form,

The formula reads: $(1+x)^n = \binom{n}{0} x^n + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$ or equivalently $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Pythagorean Theorem

The Pythagorean Theorem is a relation in Euclidian geometry among the three sides of a right-angled triangle. It states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. The theorem can be written as an equation relating the lengths of the sides a, b and c, often called the Pythagorean equation: : $a^2 + b^2 = c^2$, where c represents the length of the hypotenuse, and a and b represent the lengths of the other two sides.

The theorem is named after the Greek mathematician Pythagoras (ca. 570 BC - ca. 495 BC), who by tradition is credited with its proof although it is often argued that knowledge of the theorem predates him. Although there is little surviving proof that they applied it within a mathematical framework, there is evidence that Babylonian mathematicians understood the formula. Chinese, Indian, and Mesopotamian mathematicians have all been credited with independently discovering the solution; some have even offered proofs for certain circumstances.

The number of proofs for the theorem may be the highest of any mathematical theorem. There are both algebraic and geometric proofs, some of which date back thousands of years. The theorem can be applied to a variety of contexts, including higher-dimensional spaces, non-Euclidean spaces, non-right triangle-containing objects, and even non-triangular things altogether. n-dimensional solids, however. The Pythagorean Theorem "has attracted interest outside mathematics as a symbol of mathematical obscurity, mystique, or intellectual power; popular references are abundant in literature, plays, musicals, songs, stamps, and cartoons."

Key ideas

Key IV

- Theorem is a mathematical proposition that has been proven to be true.
- A mathematical theorem is a mathematical conjecture that has been proven
- The Pythagorean Theorem is a relation in Euclidian geometry among the three sides of a right-angled triangle

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply mathematical theorem in the teaching and learning of mathematics?
- How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?

Discussion

- How has this session equipped you to be a better classroom practitioner?
- How has your idea of mathematical theorem equipped you to be a better mathematics teacher?
- How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 6: MATHEMATICAL PARADOXES/ANTINOMY

In this session, we will focus on explaining mathematical paradox; as well as its applications in the teaching and learning of mathematics. It is hoped that learners would be able to apply mathematical paradox as used in mathematics education.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain mathematical paradox
2. apply the concept of mathematical paradox in the teaching and learning of mathematics

A proposition that is undecidable or that has been shown to be incapable of being proven to be either true or incorrect is known as an antinomy. Kurt Gödel is the mathematician whose name is associated with this; he demonstrated in 1931 that there are propositions in every mathematical system that cannot be resolved. A paradox is another name for a mathematical antinomy.

The mathematical definition of the word paradox has to do with how something is subjectively perceived, how a statement is understood, etc. A paradox is a statement that contradicts itself or a circumstance that appears to defy logic, to put it simply. It's a claim that defies common sense at times while also being profoundly obscure and profoundly profound at other times. Anything that on the surface looks to be false but is actually true, or that initially seems to be true but is actually false, or that initially seems to be self-contradictory, is a mathematical paradox. A paradox is an actual conclusion that surprises our human senses. They are in perfect health. The amazing thing about mathematics is that they do, in fact, awaken our intuition.

"Two fathers and their two sons leave town," for instance. The town's population falls by three as a result. False? True, assuming the trio consists of a grandfather, a father, and a son. Different sorts of paradox exist. One such instance, which we might refer to as a phenomenological paradox, is when fundamental truths about what the mathematics is meant to depict are contradicted by the mathematical conclusions. Then there are assertions that can be proven to be both true and untrue, known as logical paradoxes.

Think a little about this. Did you know that “0.999...” Equals “1”? Check the proof:
Let $x = 0.999\ldots$
$Then\ 10x = 9.999\ldots 10x - x = 9.999\ldots - 0.999\ldots$
$9x = 9.$
$x = 1.$
$Therefore, 0.9999\ldots = 1.$

Some examples of Antinomies

Barber's Hypothesis.
English philosopher Bertrand Russell (1872–1970) imagined a community with a male barber and all the men had clean shaven faces. It was well known that the local barber exclusively shaved the men who did not shave themselves. Only those who don't shave themselves get shaved by me. And who did the barber's shaving? Set theory was developed as a result of extremely meticulous formalisation efforts made by mathematicians. But even the most basic understanding of set theory results in the Barber's paradox. He was shaved by the barber if the barber did not shave himself at that point—a paradox. Another contradiction is that if the barber shaves himself, then he was not shed by the barber.

Libel of Epimenides
The name of this logic game pays homage to the Cretan philosopher of the sixth century BC. Epimenides of Knossos is quoted as saying, "Cretans are always liars. Titus 1:12, when the author claims regarding Cretans that "they are all liars, as one of their own has said," mentions

Epimenides. It is a paradox within itself. This is supposed to surface when the veracity of Epimenides' claims are questioned. I lie all the time, but are they telling the truth or me?

Aristotle's Paradox of Place:

"If everything that exists has a space, then place too will have a place, and so on endlessly."

The Galileo Paradox

A demonstration of one of the unexpected characteristics of infinite sets can be found in Galileo's paradox. Galilei made assertions about the positive integers that appear to be in conflict. First of all, certain numbers have squares while others do not; therefore, the total number of numbers, both squares and non-squares, must be greater than the sum of the squares alone. Yet every square has a unique positive number that serves as its square root, and every number has a unique square as well. As a result, none can exist in excess of the other. In the setting of infinite sets, this is an early use of the concept of one-to-one correspondence. Galileo came to the conclusion that infinite sets are not subject to the concepts of less, equal, and larger.

Using the same techniques, Cantor demonstrated that this constraint was unnecessary in the nineteenth century. It is possible to meaningfully define comparison across infinite sets, and by this definition some infinite sets are strictly larger than others (by which definition the two sets he discusses, integers and squares, have "the same size").

Achilles and the Tortoise

According to Aristotle, the slower runner must always keep a lead since the fastest runner can never pass the slower one because the pursuer must first get to the starting line. Achilles and the tortoise are competing in a footrace in the Achilles and the Tortoise dilemma. Achilles, for instance, gives the turtle a 100-meter head start. Achilles will reach the tortoise's starting place after a limited amount of time if we assume that each racer starts running at a constant speed (one very fast and one very slow). The tortoise has only travelled a significantly shorter distance during this time—say let's 10 metres. Achilles will then need additional time to go that distance, by which point the tortoise will have advanced farther. Additionally, more time will be needed to arrive at this third location as the tortoise continues to move forward. Achilles always has further to go after passing where the tortoise has been. Achilles can never catch up to the tortoise because there are an unlimited number of spots he must travel to where it has already been. This falls under the category of a motion paradox.

Lazy-bones Paradox

Isn't it pointless, for instance, to visit a doctor if destiny has a master plan that details everything that will occur? If I am ill and it is in my nature to get better, then whether or not I see a doctor, I will get better. A doctor cannot help me if it is my destiny to never recover my health. How could you contest the viewpoint that was expressed?

The Law Professor and his alumnus

A Greek teacher gave his pupils legal training. However, he only requested payment from a student after winning his first case. One pupil never took up a case after quitting the teacher.

In the end, the lawyer filed a lawsuit against the student, claiming that if the judge ruled in her favour, she should be reimbursed for her legal fees. If the judge rules in my student's favour, he will have won his first lawsuit and will be required to pay my legal expenses. If the judge rules in my favour, I won't be required to pay fees, the student contended. If it rejects my arguments, I will have lost my first lawsuit and won't be required to pay any fees moving forward.
Who is correct?

UNIT 3: HISTORICAL DEVELOPMENT OF MATHEMATICS

The concepts axioms, operations, algorithms, proofs, conjectures, theorems, and antinomies were covered in detail in unit 2 of the course. We are all aware with the process in which guesses or conjectures are tested and retested in order to produce mathematical proofs that produce mathematical theorems. This lesson will teach us about the development of numeration and number systems throughout history.

The Pythagorean universe was dominated by numbers. Integers and natural numbers held sway throughout this time. The world was still governed by numbers. The natural numbers are the foundation of all mathematics. Children are aware of object-related numbers. Numbers written as symbols are referred to as numerals.

Every culture created its own concepts of number. Long before recorded history began, the idea of numbers and the practise of counting were already well-established. The first mathematical activity was counting. Even in the most prehistoric eras, man had some understanding of numbers. They could at least roughly track when objects were added to or removed from a group. Later, it became crucial to know how many people were in a family, how many flocks one possessed, etc. With the use of fingers, notches in wood, scratches on stones, and knots in strings, rudimentary tally methods based on the one-to-one correspondence concept are likely where counting first originated. Later, a word tally against the number of items in a small group was established as a vocal sound. These numbers eventually developed a variety of symbols. For instance, there are numerous ways to write the number six, including 6, VI, six, ~~////~~ /, etc.

These symbols all stand for the same idea, despite their differences. Numbers are the symbols used to symbolise the idea of "sixness," also known as a concept or idea. It took a long time to achieve the abstraction of a common feature of "sixness," represented by some sound, when viewed separately from any specific associations. We may now feel as though we have forgotten the connection between sets of concrete objects and the number words. We have a variety of historical systems, including those used by the Egyptians, Babylonians, American Indians, the Vigesimal scale (based on 20), and the highly advanced Mayans. Gaelic, Danish, Welsh, Greenlandic, Roman, and Hindu-Arabic numeral systems all have traces of it.

Learning outcomes

You should be able to:

1. explain numbers and numeration systems by the end of the unit.
2. describe the key characteristics of the Egyptian numbering system;
3. describe the key characteristics of the Babylonian numbering system;
4. describe the key characteristics of the Roman and Hindu-Arabic numbering systems;
5. identify and explain the fundamental characteristics of natural numbers;
6. differentiate between the various Pythagorean (and figurative) numbers; and
7. describe some characteristics of integers.

SESSION 1: NUMBER AND NUMERATION SYSTEMS

We will explain numbers and the numeration system in this session, as well as how they are used in mathematics instruction. In the teaching and learning of mathematics, it is intended that students would be able to use their understanding of number and numeration.

Learning outcomes

The participant will be able to:
By the end of the session, the participant will be able to:

1. explain number and numeration system
2. explain the relationship between number and numerals
3. apply the concept of numeration system in the teaching and learning of mathematics

The natural numbers are the foundation of all mathematics. Every culture created its own concepts of number. Long before recorded history began, the idea of numbers and the practise of counting were already well-established. The first mathematical activity was counting. Man had some understanding of numbers even in the most prehistoric ages. At least they were able to roughly indicate when things were added to or removed from the group. Knowing how many people made up a family, how many flocks someone possessed, etc., became crucial later on. With the use of fingers, notches in wood, scratches on stones, and knots in strings, rudimentary tally methods based on the one-to-one correspondence concept are likely where counting first emerged. Later, a word tally against the number of items in a small group was established as a vocal sound. These numbers eventually developed a variety of symbols. For instance, there are numerous ways to write the number seven, including 7. VII, ~~////~~ //, etc.

These symbols all stand for the same idea, despite their differences. The written symbols used to symbolise the concept of "sevenness," also known as a number, are the numerals. It took a long time to achieve the abstraction of a general attribute of "sevenness," represented by any sound, when viewed separately from any concrete associations. We may now feel as though we have forgotten the connection between sets of concrete objects and the number words.

A numeration system is a collection of numerals that has been systematically arranged. A numeration system consists of a collection of fundamental symbols and a set of rules for generating additional symbols from them. One of the greatest human achievements was the development of a practical system that allowed us to pass on knowledge from one generation to the next. The development of symbolic representation takes many years. There will undoubtedly be more numeration systems created in the future.

As a "simple grouping system," the written numeration system was first developed. By organising the numbers into practical core categories, the counting process had to be systematised as it became more broad. The size of the group was decided by the matching method used. Names were given to the numerals 1, 2, 3,..., b after a base (radix or scale) of some number, b, was chosen. The names for numbers greater than b were then created by combining the names of the smaller numbers. The majority of early number systems were based on the fingers of one or both hands, as seen by the prevalence of the numbers 5 and 10 in modern number systems. Thus, according to philologists, the number 11 is "ein lifon," which means "one left over" or "one over ten," the number 12 is "twe lif," which means "two over ten," the number 13 is "three and ten," etc., the number 20 is "twe-tig," which is two tens, the number 21 is "two tens," and the number one and 100 denotes ten times ten.

Numerous ancient numeration systems have been preserved, including those used by the Egyptians, the Babylonians, the Mayans, the Romans, the Vigesimal scale (based on 20), the American Indians, and the Romans. Traces of these systems can also be found in Gaelic, Danish, Welsh, and Greenlandic.

Key ideas

Key I

- Counting probably started with simple tally method on the principle of one-to-one correspondence- use of fingers, notches in woods, scratches on stones, knots in strings.
- The concept or idea of "sevenness" is called a number, the written symbols used to represent the concept are the numerals.
- A numeration system is a set of basic symbols and some rules for making other symbols from them

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of number and numeration system in the teaching and learning of mathematics?
- How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?

Discussion

- How has this session equipped you to be a better classroom practitioner?
- How has your idea of number and numeration system equipped you to be a better mathematics teacher?
- How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 2: EGYPTIAN NUMERATION SYSTEM

In this session, we will focus on explaining Egyptian numeration system; as well as its applications in the teaching and learning of mathematics. It is hoped that learners would be able to apply their concepts of Egyptian numeration system in the teaching and learning of mathematics.

Learning outcomes

By the end of the session, the participant will be able to explain the:

1. Egyptian numeration system
2. two primary sources of Egyptian numeration system

The Egyptians were one of the first cultures to be known to use numerals. By the time of the first dynasty, or around 2850 BC, the Egyptians had such a system in place. The ancient writing system known as hieroglyphics used by the Egyptians included symbols for mathematics. The Egyptians employed mathematics in a wide range of applications, including the use of astronomy to calculate dates for religious celebrations and to forecast the yearly flooding of the Nile.

Egyptian mathematics can be found in two primary sources and numerous secondary sources. The Rhind (or Ahmes) Papyrus and the Moscow Papyrus are the principal sources, and together they offer 112 real-world math and geometry problems and their solutions. Three papyri from around 1800 BC are among the secondary sources: the Egyptian Mathematical Leather Roll (which contains a table of 26 decompositions of unit fractions), the Berlin Papyrus (which contains two problems of simultaneous equations, one of which is of the second degree), and the Reisner Papyrus (volume calculations).

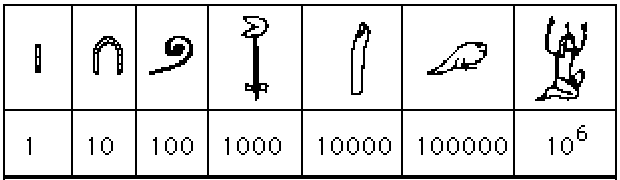
The main source of our knowledge of ancient Egyptian mathematics is the Rhind Mathematical Papyrus (RMP), also known as the Ahmes Papyrus. It is an excellent illustration of Egyptian mathematics. It bears the name of Scottish antiquarian Alexander Henry Rhind, who bought the papyrus in Luxor, Egypt, in 1858. It was reportedly discovered during unauthorised excavations inside or close to the Ramesseum. It was created in 1650 BC. The majority of the papyrus is currently housed in the British Museum, which purchased it in 1864.

The Rhind Mathematical Papyrus was produced in Egypt during the Second Intermediate Period. The scribe Ahmes copied it from a now-lost manuscript from King Amenenhat III's reign (12th dynasty). This text, which is nearly 5 metres long in total and is written in the hieratic script, is 33 cm tall.

The Rhind papyrus' first section is made up of reference tables and a set of 20 math and 20 algebraic puzzles. Simple fractional expressions are used to begin the problems, which are then followed by completion (sekhem) issues and more complex linear equations (aha problems). The Rhind papyrus' second section is made up of "mensuration problems," which are geometrical puzzles. The Moscow Mathematical Papyrus is older than the Rhind Papyrus, but the former is larger.

The Golenischev Mathematical Papyrus, often known as the Moscow Mathematics Papyrus (MMI), was copied by an unidentified scribe (-1850 BC). About 25 practical mathematics problems (basic equations) and their solutions can be found in the Moscow papyrus. V S. Golenishchev (d. 1947) bought it and sold it to the Moscow Museum of Fine Art. It is 3 inches broad and 15 feet long. Problem 14 dealt with a frustum's volume. One is instructed by the scribe to square the numbers two and four and to add the product of those squares to the total of those squares. Multiply this by the sixth-third power. Verify the number is 56.

Like the modern system, the Egyptian numbering system was similarly based on units of 10, but rather than using a number's position to determine its value, the Egyptians employed various images to symbolise various units of 10. The hieroglyphic symbols for 1, 10, 100, 1000, 10,000, 100,000, and one million:



Egyptian numeral hieroglyphs

The Egyptian system was based on the **simple grouping** system. Their symbols were used **additively** to express any number. Each symbol is repeated the required number times, but no more than **nine** repetitions. For example, 432 was written as эээээ∩∩∩II or э∩э∩II∩ээ or ∩IээՈэI∩э

It was simply based upon addition. They just needed to add up the numbers represented by the symbols. The position of the individual symbols is not important.

The system was not a **place value** system. It had no symbol for zero. The Egyptians did repetitive type of arithmetic. They performed addition and subtraction by repeating the symbol and by regrouping. Eg.

э∩∩∩∩IIIIIIIII + ээээ∩∩∩II

= ээээээ∩∩∩∩∩∩∩∩I

Key ideas

Key I

- Egyptian mathematics symbols were part of their ancient writing system called hieroglyphics
- There are two primary sources and a number of secondary sources on Egyptian Mathematics.
- The **Rhind Mathematical Papyrus** (RMP) which is also known as the Ahmes Papyrus, is the major source of our knowledge of the mathematics of ancient Egypt
- **Moscow Mathematics Papyrus** (MMI), also known as the Golenischev Mathematical Papyrus was copied by an unknown scribe (-1850 BC)

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of Egyptian numeration system in the teaching and learning of mathematics?

• How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of Egyptian numeration system equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 3: BABYLONIAN NUMERATION SYSTEM

In this session, we will focus on explaining Babylonian numeration system; as well as its applications in the teaching and learning of mathematics. It is hoped that learners would be able to apply their concepts of Babylonian numeration system in the teaching and learning of mathematics.

Learning outcomes
By the end of the session, the participant will be able to:

3. explain Babylonian numeration system
4. explain the two primary sources of Egyptian numeration system

The ancient Mesopotamian (Babylonian) numeration system was distinct from the Egyptian system in a number of ways. The Babylonians adopted a considerably more practical positioning system than the Egyptians, who used a straightforward grouping system. Cuneiform, a well-preserved style of writing on clay tablets that is less artistic and uses wedge-shaped markings, was utilised by the Babylonians. Only two wedge-shaped characters, dating to around 3000 BC, were used.
Which are:𒁹 for 1 (one), 𒌋 for 10 (ten)

Thus 24 is written as 𒌋 𒁹𒁹𒁹𒁹
The system is repetitive from **1** through **59**. Here, the position of the symbols was important. The symbol for ten must appear to the left of any ones to represent numbers less than 60. For numbers larger than 60, the symbol for ten and the symbol for one are to the left of symbol for ten, and any symbol to the left of the symbol for ten have a value 60 times their original value. For example,

𒁹𒁹𒁹 𒌋𒌋 𒁹𒁹𒁹𒁹𒁹 $= 3(60) + 25 = 180 + 25 = 205$
𒁹𒁹 𒌋𒌋𒌋𒌋 𒁹𒁹𒁹𒁹𒁹𒁹 $= 2(60) + 46 = 120 + 46 = 166$

𒌋𒌋𒁹𒁹𒁹𒌋𒌋𒌋𒌋𒁹𒁹𒁹𒁹𒁹
$= 23(60) + 45 = 1380 + 45 = 1{,}425.$

Sexagesimal system is another name for the Babylonian system. Since numbers are written using a straightforward grouping scheme within each of the fundamental 60 groups, it is not entirely positional. It lacked the sixty various symbols. There were several uncertainties since they lacked a symbol for zero. The Babylonians went one step farther and covered any number to the left of the second group of 60 by covering 602. When measuring time in minutes and seconds, the system is still in use.

Key ideas

Key I
• The Babylonian numeration system is also called *sexagesimal system*
• There are two primary sources and a number of secondary sources on Egyptian Mathematics.
• The Babylonians used a well-preserved and less pictorial clay tablet writing using wedge-shaped marks known as **cuneiform**
• They employed only two **wedge-shaped** characters, which date from about 3000BC
• Babylonian numeration system is repetitive from **1** through **59**

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of Babylonian numeration system in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of Babylonian numeration system equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 4: ROMAN NUMERATION SYSTEM

In this session, we will focus on explaining Roman numeration system; as well as its applications in the teaching and learning of mathematics. It is hoped that learners would be able to apply their concepts of Roman numeration system in the teaching and learning of mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain Roman numeration system
2. explain the two primary sources of Roman numeration system

The Roman numeration system used the following basic symbols I for one (1), 'X' for ten (10), 'C for hundred (100), and 'M', for one thousand (1000). These are augmented by 'V' for 5, 'L' for 50, and 'D' for 500. The principle of subtraction is used in a way that does not require an additional symbol. When a smaller unit is placed before a symbol for a larger unit, it means the difference of the two units. This shows the **subtractive** principle. For example, we have IV for 4; IX for 9; XL for 40; XC for 90; CM for 900. Thus, 99 may be written as XCIX in Roman numerals and 1944 MDCCCCXXXXIIII in ancient times, but now MCMXLIV. This means that representation of a number in Roman Numeration is not unique.

The Romans achieved nothing of importance in mathematics. Roman numerals, which were difficult to calculate with, served as the foundation for the Roman numeration system. Despite this shortcoming, some European nations used the Roman numeration method for bookkeeping for another century and until as late as 1600. It is still mostly used on title pages and monuments.

Key ideas

Key I

- Roman numeration system used the following basic symbols I for one (1), 'X' for ten (10), 'C for hundred (100), and 'M', for one thousand (1000).
- Roman numeration system is augmented by 'V' for 5, 'L' for 50, and 'D' for 500
- A smaller unit placed before a symbol for a larger unit, means the difference of the two units.

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of Roman numeration system in the teaching and learning of mathematics?
- How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?

Discussion

- How has this session equipped you to be a better classroom practitioner?
- How has your idea of Roman numeration system equipped you to be a better mathematics teacher?
- How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 5: HINDU-ARABIC NUMERATION SYSTEM

In this session, we will focus on explaining Hindu-Arabic numeration system; as well as its applications in the teaching and learning of mathematics. It is hoped that learners would be able to apply their concepts of Hindu-Arabic numeration system in the teaching and learning of mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain Hindu-Arabic numeration system
2. explain the two primary sources of Hindu-Arabic numeration system

Ten symbols make up the Hindu-Arabic numeration system (may be because we have ten fingers). These are: 0, 1, 2, 3,,,,,,,,,,, These figurines' earliest known incarnations are said to have come from India around 300 BC. It may have been created by Hindus and spread to Western Europe by Arabs. The earliest surviving examples of the numbers were discovered on some stone columns built in India by King Asoka around 250 BC. The zero was not present in the ancient numerals; it probably came from Babylon through Greece and India. Indians are credited as being the first people to understand that zero is both an integer and a placeholder. The zero symbol and the concept of the positional system had reached Baghdad by 750 AD and had been translated into Arabic. Such a finished Hindu system was detailed in a text written in 825 AD by the Persian mathematician al-Khowarizmi.

In the 18th century, it is known that these numerals travelled from Spain to Europe. The first academic in Europe to instruct these numerals was Gerbert (after known as Pope Sylvester II), who studied in Spain. These numerals are known as Hindu-Arabic numerals due to their origin. It also goes by the name decimal numeration system due to the use of 10 fundamental symbols. The system was introduced to Europe by the Arab mathematician al-Khowarizmi in a treatise titled "Liber Algorisms de Numero Indorum."

The system's introduction was not without criticism. The "Algorists," who supported the system, and the "Abacists," who supported the status quo—using Roman numerals and performing calculations on an abacus—were the two opposing factions. This 400-year conflict between the Algorists and the Abacists took place. The Roman Catholic Church supported the abaconists. Roman numerals, they claimed, were simpler to write, memorise, add to, and subtract from than Hindu-Arabic numerals. The Chinese suan pan and the Japanese soroban are two highly developed abacus types still in use today.

However, the Hindu-Arabic system has lasted ever since because:

(a) It uses ten symbols only;

(b) Larger numbers are expressed in terms of powers of ten; there is no limit to the size of the numbers that can be written using only the digits from 0 through 9.

(c) It is positional.

For positional systems, after the base b has been selected, basic symbols called **digits** are adopted for 0, 1, 2, 3, -- b-1. Thus, a number A can be written uniquely in the form

$A=a_nb^n+a_{n-1}b^{n-1}+a_{n-2}b^{n-2}+\ldots..+a_2b^2+a_1b^1+a_0,$
where 0 < a <b, i=0, 1, 2..., n.
Thus, the number A in the base b can be represented by the sequence of basic symbols: $a_na_{n-1}a_{n-2}a_{n-3}\ldots a_2a_1a_0$.

For example, 7354 in base ten is written $(7 \times 10^3) + (3 \times 10^2) + (5 \times 10) + 4$.

A number written in this form is said to be in the expanded form. Thus, in the number 7354, the 7 stands for 7(1000), 3 for 3(100), 5 for 5(10) and 4 for 4(1).

The symbol for zero is used to indicate any missing powers of the base. Obviously, the symbol for zero become a great convenience in the positional numeration system. The positional numeration system is a logical outgrowth of the **multiplicative grouping system**.

A period called 'decimal point in the decimal system is used to separate the fractional parts from the whole number part. E.g., 8537. 453 may be expanded as $857.453 - 800 + 50 + 7 + 0.4\ 0.05 + 0.003$
That is, $857.453 = 800 + 50 + 7 + 0.4 + 0.05 + 0.003$
Thus, $857.453 = (8x102) + (5x10) + (7x1) + (4x10 - 1) + (5 \times 10 - 1) + (3 \times 10 - 3)$

Other Bases
If the base is less than 10 $(b < 10)$, we may use our ordinary base ten digits. For example, we may express 6543 in base seven using the basic symbols 0, 1, 2, 3, 4, 5 and 6 and write it as 6543_{seven}. To convert this number to base ten we expand and simplify.

That is, $6543even = 6(73) + 5(72) + 4(7) + 3 = 2{,}334ten$. If the base is greater than 10 $(b > 10)$, we need to augment our digit symbols by some new basic ones. Thus, for b = 12 we have 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e for our basic symbols where t and e stand for ten and eleven respectively.
Thus $(5t4e)_{eleven}$ can be expressed as:

$$(514e)twelve = (5x123) + (t\,x\,122) + (4 \times 121) + (e\,x\,120)$$

$$= (5x1728) + (10x144) + (4 \times 12)\,(11 \times 1)$$
$$= 8640 + 1440 + 48 + 11$$
$$= 10{,}139$$

Base 12 system is also called duodecimal system.
In general, changing from base ten (ordinary scale) to a base b follows the following process:
Let N be the number. Then, $N=a_na_{n-1}a_{n-2}a_{n-3}\ldots a_2a_1a_0$.
We then determine the integers
$a_n, a_{n-1}, a_{n-2}, a_{n-3}, \ldots a_2, a_1, a_0$, in the expression:
$N=a_nb^n+a_{n-1}b^{n-1}+a_{n-2}b^{n-2}+a_{n-3}b^{n-2}+\ldots+a_2b^2+a_1b^1+a_0$.
Where $0 \le a_i < b$

We then divide this polynomial repeatedly by the base, b, and note the remainders. Dividing N by the base b we have

$\frac{N}{b}=a_nb^{n-1}+a_{n-1}b^{n-2}+a_{n-2}b^{n-3}+\ldots..+a_2b+a_1+\frac{a_0}{b}$

The remainder, a_0 of this division becomes the last digit in the desired representation. Dividing N^* by b, we have

$\frac{N^*}{b}=a_nb^{n-2}+a_{n-1}b^{n-3}+a_{n-2}b^{n-4}+a_{n-3}b^{n-5}+\ldots\ldots+a_2+\frac{a_1}{b} = N^{**}+\frac{a_1}{b}$

and the remainder, a_1 becomes the next to the last digit in the desired representation. This procedure continues successively to obtain all the digits $a_0, a_1, a_2, \ldots\ldots$ This is systematized as in the specific example in expressing 497 as a base five numeral,

5 divides	497	Remainder
5 divides	99	2
5 divides	19	4
5 divides	3	4
5 divides	0	3

We write the remainders, starting from the last one. This yields the result $497=3442_{\text{five}}$.

OR, $497 = 5\,x\,99 + 2$
$99 = 5\,x\,19 + 4$
$19 = 5 \times 3 + 4$
$3 = 5 \times 0 + 3$

Key ideas

Key I

- Hindu -Arabic numeration system uses ten symbols
- Hindu -Arabic numeration system is also called decimal numeration system because ten basic symbols are used.

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of Hindu-Arabic numeration system in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of Hindu-Arabic numeration system equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 6: BASIC PROPERTIES OF NATURAL NUMBERS

In this session, we will focus on explaining Hindu-Arabic numeration system; as well as its applications in the teaching and learning of mathematics. It is hoped that learners would be able to apply their concepts of Hindu-Arabic numeration system in the teaching and learning of mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain the properties of binary operations
2. explain the very types of numbers
3. apply the concepts of primary factorization
4. Explain integers

1. Closure Property for addition and multiplication: When we add (or multiply) any two natural numbers we will obtain a natural number. That is, if a, b ε N then a + b=c ε N and a x b =k ε N.

2. Property of Order

a) **The Commutative Property** for addition states that the order in which two numbers are added makes no difference. $Le\ a + b = b + a;\ a, be\ N$. (To travel back and forth from home to work).

The Commutative property for multiplication states that the order in which two numbers are multiplied makes no difference. a x b = b x a, a, b ε N. (Three groups of five oranges give the same result as five groups of three oranges).

b**) The Associative Property**: The order of adding or multiplying three or more numbers does not affect the result. In adding three or more numbers, the associative property states that no matter which two numbers are added first, the final result is the same. i.e $(a + b) + c = a + (b + c)\ a + b + c$.. In multiplying three or more numbers, the associative property states that no matter which two numbers are multiplied first, the final result is the same. That is. $(a\ x\ b)\ x\ c = a\ x\ (b\ x\ c) = a\ x\ b\ x\ c$

The associative property allows us to group numbers for addition and multiplication. The parentheses indicate the numbers to be added or multiplied first.

3. **Distributive Property of Multiplication over Addition and Subtraction**

Suppose there are 9 families each having 3 males and 4 females. There are two ways of finding the total number of people in all the six families.

Method 1: First, find the total number of males and the total number of females in the 9 families separately and then later add the two results. This gives (9 x 3) males + (9 x 4) females. That is, $27 + 36 = 63$, giving 63 people in all.

Method 2: First, find the total number of people in each family and then add all.

Each family has (3 + 4) people and for the 9 families we have $9 \times (3 + 4) = 9x7 = 63$. Since the two results are the same we can conclude that: $9\ x\ (3 + 4) = (9 \times 3) + (9 \times 4)$

The Distributive property states that for any $a, b, c \in N, a\ x\ (b + c) = (a\ x\ b) + (a\ x\ c)$

Or, $a\ x\ (b + c) = (ab) + (ac)$, and

$$a\ x(b - c) = (a\ x\ b) - (a\ x\ c)\ Or, a\ x\ (b - c) = (ab) - (ac)$$

The Distributive property allows us to simplify arithmetic such as:

$$9\ x\ 73 – 9\ x\ (70 + 3) = (9\ x\ 70) + (9\ x\ 3) = 630 + 27 = 657.$$

Or, as

$$9\ x\ 73 - 9\ x\ (80 - 7) = (9x80) - (9x7) = 720 - 63 - 657$$

Odd and Even Numbers

The set of natural numbers can be split into two categories, even and odd.

Even numbers are at every other position in the sequence of natural numbers. They leave no remainder when divided by 2. These are: 2, 4, 6, 8,... Counters/objects representing these numbers can be put into pairs.

Odd numbers start from the first natural number and every other number in the sequence. They leave a remainder of 1 when divided by the first even number, 2. These are 1, 3, 5, 7, Counters/objects representing these numbers cannot be put into pairs.

Prime and Composite Numbers

Every counting number greater than 1 has at least two distinct divisors, itself and one. Classifying the counting numbers according to the number of divisors each has leads to the following definition:

A **prime** number is a counting number that has exactly two divisors. Any positive integer greater than 1 which has no factors apart from 1 and itself is a prime number. The counting numbers that have more than 2 divisors are called composite numbers. One is neither a prime nor a composite number, why? Try to explain.

The sequence of prime numbers is 2, 3, 5, 7, 11, 13, 17, 19,... Only the first prime number is even, all others are odd. The sequence is highly irregular and there is no iterative method for producing the next in the sequence.

Eratosthenes' Method of Finding Prime Numbers

A Greek Mathematician named Eratosthenes first used a technique called the Sieve of Eratosthenes (more than 2000 years ago) to find the prime numbers smaller than some given number. That is, a method for removing the composite numbers from the set of natural numbers, leaving the prime numbers. The process is as follows:

1. List all the counting numbers up to the given number, say 100
2. Cross out 1, since it is not classified as a prime
3. Draw a circle around 2, the smallest prime number. Cross out every following multiple of 2.
4. Draw a circle around 3, the next prime number. Then cross out each succeeding multiple of 3.
5. Circle around the next open number, 5 and cross out all

subsequent multiples of 5.

6. Circle around the next open number, 7 and cross out all subsequent multiples of 7.Since 7 is the largest prime number less than 100, we end the process and list all numbers left as prime numbers In general, to find the prime numbers less than a natural number N.

- Find the largest prime less than or equal to $\sqrt{N}$
- Cross out the multiples of primes less than or equal to $\sqrt{N}$
- All the remaining numbers in the chart are prime numbers.

Prime twins (or twin primes) are pairs of consecutive odd numbers that are primes and differ by 2. Examples are 3 & 5,5 & 7, 11 & 13, 17 & 19, 29 & 31, etc. The largest known twin primes were 1,000,000,009,649 & 1,000,000,009,651 found in 1986.

A **prime triplet** is a set of three prime numbers that differ by 2. The only known set is 3, 5 & 7

3.3 Application of Prime Factorization

Every composite natural number can be expressed as a product of primes. Except for the order of the factors, this expression is unique. For example,

$12 = 2\times2\times3 = 2^2 \times 3$; $54 = 2\times3\times3\times3 = 2\times3^3$

Prime numbers form the building block of natural numbers. That is, Z= $a_1a_2a_3\ldots a$;
where a is a prime, k= 1, 2, 3,.....

The Highest Common Factor (HCF) or the Greatest Common Divisor (GCD) of two or more given natural numbers is the greatest number which is a factor of the given numbers.

A To find the **HCF** of numbers:

1 Find the prime factorization of each number.
2 Write it in canonical (index) form.

3 Choose the representative of each factor with the smallest exponent.

4 Take the product of the representatives as the HCF.

Example 1

Find the HCF of 96 and 300

Solution

$96 = 2\times2\times2\times2\times2\text{x}3 = 2^2 \text{ x } 3$

$300 = 2\times2\times3\times5\text{x}5 = 2^2 \text{ x}3\text{x}5^2$

Representatives with the smallest exponent: $2^{\ 2}$ x 3. The product of 2^2 x 3 is 12. Therefore, HCF of 96 and 300 is 12.

Thus 12 is the highest number that can divide 108 and 300. The greatest number which is a factor of 108 and 300 is 12. Lowest Common Multiple (LCM) of two or more natural numbers is the smallest number into which the given numbers can divide i.e. the least number that is divisible by the given numbers.

To find the LCM of numbers:
1 Find the prime factorization of each number.
2 Write it in canonical (index) form.
3 Choose the representative of each factor with the greatest exponent.
4 Take the product of the representatives as the LCM.

Example 2
Find the LCM of 96 and 300.
Solution

$96 = 2×2×2×2×2x3 = 2^5$ x 3
$300 = 2×2×3×5x5 = 2^2 × 3 × 5^2$
Representatives with the highest exponent: 2^5 x 3 x 5^2.
The product of 2^5 x 3 x 3^2 is 2400.
Therefore, LCM of 96 and 300 is 2,400.
Thus 2,400 is the smallest number that 96 and 300 can divide.

3.4 Integers
The primitive agricultural-type society needed only the natural numbers N={1, 2, 3, 4,} When zero was discovered and annexed to the set of natural numbers we have the set of Whole numbers, W={0, 1, 2, 3, 4,}. This satisfied society for several thousands of years. Bookkeeping later advanced as society evolved and problems such as 3-5-2 arose which could not be solved using the whole numbers only. The opposites of the natural numbers were introduced and annexed to the set of whole numbers to obtain the set of integers, Z = -3, -2, -1, 0, 1, 2,3,)

The negative numbers were developed quite late. The Chinese had some knowledge of negative numbers as early as 200BC and in the 7th Century AD the Hindu Brahmagupta stated the rules for operations with positive and negative numbers. Chinese represented negative numbers by putting them in red (*compare with the present day accounting)*. The Hindus represented them by putting a circle or dot over the number. However, as late as the 16th century, some European scholars were referring to numbers such as (zero minus one (0 - 1) as "absurd". In 1545, an Italian scholar, Cardano, in presenting a paper on the elementary properties of negative numbers referred to integers as "f**ictitious** numbers". The positive numbers he referred to as "true" numbers. The word integer was derived from "numbers with i**ntegrity"**

Note all integers (except zero) consist of two parts - the signed part and the whole number part. An absolute value gives the numerical value of a quantity independent of direction or sign. The absolute value of x is its undirected distance from zero (on the number line). We use the symbol]x[; and]x[=-x, when x is negative or]x[when x is positive.

Multiplication of Integers.
We can demonstrate practically that a x b = ab and that -a x b = ab and that -a x b = -ab**; interpreting multiplication as repeated addition**. We can develop multiplication of two negative numbers from these two facts.

Example:

-3 x 5 = -15	-4 x 5 = -20	-5 x 5 = -25
-3 x 4 = -12	-4 x 4 = -16	-5 x 4 = -20
-3 x 3 = -9	-4 x 3 = -12	-5 x 3 = -15

Pause and ponder!!!!! Can you predict the next lines?

-3 x 2 = -6	-4 x 2 = -8	-5 x 2 = -10
-3 x 1 = -3	-4 x 1 = -4	-5 x 1 = -5
-3 x 0 = 0	-4 x 0 = 0	-5 x 0 = -5
-3 x -1 = 3	-4 x -1 = 4	-5 x -1 = 5
-3 x -2 = 6	-4 x -2 = 8	-5 x -2 = -10
…………	………….	…………

Observe from the pattern that in the first case the second factor decreases by 1 from 5 to 0 while the product consistently increases by 5 from -25 to 0. Following the pattern, we should expect the next second factor to be -1 and the next product to be 5 more than zero and so should be 5, giving -5x-1=5 indicating a product of two negative numbers yielding a positive number. The next product. -5 x -2= 10 also yielded a positive number.
Observing the pattern in the remaining two cases for -3 x -1 =3 and -4 x -1 = 4 supports the conclusion that the product of two negative integers gives a positive integer. In general, we say that -a x -b = ab, where a and b are integers.

3.5 Pythagoreans

The Pythagoreans studied numbers for various purposes. The inherent union between geometry and arithmetic became clear when the Pythagoreans discovered the theorem - in any right triangle, the sum of the squares built on the legs is equal to the square built on the hypotenuse. This gave birth to what is often called the Pythagorean triplets, some examples of which are (3, 4, 5), (5, 12, 13), and (7, 24, 25). These triplets satisfy the theorem stated about the length of sides of a right-angled triangle. The special case of an isosceles right-angled triangle with the length of legs being unity (1) gives rise to the non-existence of the square root of 2 (12) among rational numbers and hence illustrating irrational numbers.

One other reason why the Pythagoreans studied numbers was to find certain mystical properties in them. They called the odd numbers as "masculine" numbers and the even numbers as "feminine" numbers. They also referred to some numbers as amicable, deficient, perfect and abundant numbers.

Perfect Numbers: A **Perfect** number is a counting number that is equal to the sum of all its divisors that are less than the number itself. The divisors of a number that are less than the number itself are called proper divisors. For example, the proper divisors of 6 are 1, 2 and 3 and 1+ 2+3=6. Since the sum of the proper divisors of 6 is 6, we say 6 is a perfect number. Equivalently, a perfect number is a number that is half the sum of all of its positive divisors excluding the number itself.

Perfect numbers are rare and until recently only few had been found. Verify that 28 and 240 are perfect numbers and find some more. All even numbers that are perfect numbers are of the form $2^{(p-1)}(2^p-1)$, where p and (2^p-1) are prime numbers. It is not known yet if an odd one exists. The 27th perfect number is $2^{44,496}(2^{44,497}- 1)$. It has 25,000 digits.

Abundant Numbers: An **Abundant** number has the sum of its proper divisors greater than the number itself. For instance, 24 is an abundant number because the proper divisors are 1, 2, 3, 4, 6 and 12 and the sum of these divisors is 36 which is greater than 24.

Deficient Numbers: If the sum of the proper divisors of a given number is less than the number, the number is said to be deficient. The number 8 has proper divisors 1, 2 and 4 and the sum is 7 which is less than 8. Therefore, 8 is a deficient number.

Amicable or Friendly Numbers: Two numbers are said to be **amicable or friendly** if each is the sum of the proper divisors of the other. They were used as friendly charms, in astrology and sorcery. For example, 220 and 284 are friendly numbers because, Proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 with sum 284 Proper divisors of 284 are 1, 2, 4, 71, 142 with sum 220.

In 1636, Fermat discovered the pair, 17,296 & 18,416. Check to verify.
Find some more.

Polite numbers: Polite numbers are natural numbers that can be expresses as a sum of two or more consecutive natural numbers. Some examples are 9, 11, 18.
Eg. 9 = 2+3+4 or 4+5; 11 = 5+ 6; and 18 = 3+4+5+6 or 18-5+6+7.

Find some more.

Other **figurative numbers** are:

Square numbers - these are formed by arranging counters to form sides of squares. The numbers are 1, 4, 9, 16, ... They are numbers that have 2 equal factors.

Triangular numbers - these are formed by arranging counters to form the sides of a triangle. The numbers are 1, 3,6,10,15, ...

The relationship between square numbers and triangular numbers is that the sum of two consecutive triangular numbers gives a square number. E.g., 1+3-4 which is the 2 square number while 1 and 3 are the 1st and 2nd triangular numbers and 6 10 16 which is 4th square number while 6 and 10 are the 3rd and 4th triangular numbers
Pentagonal numbers can be formed by arranging counters to form pentagons. The numbers are 1, 5, 12, 22, 35,

3.6 Development of Algebra
In mathematics, the word algebra is a structure or a set of axioms that form the basis for what is accepted and what is not, when manipulating symbols of that system The basic understanding of an algebra is slated to what is called a mathematical system. A **mathematical system** is a set with at least one defined operation and some developed properties

The word "algebra" is derived from the Arabic word "al- jabr" which means restoration or completion. Algebra is important because it is one of the largest, broadest and most relevant type of mathematics today. Algebra provides a generalization of arithmetic by using symbols, usually letters, to represent numbers. For example, it is obviously true that $2+3 = 3+2$

This arithmetic statement can be generalized using algebra to $x + y = y + x$, where x and y can be any number.

Algebraic Equations

An algebraic equation shows the relationship between two or more variables. The equation, $A = \pi r^2$ states that the area (A) of a circle equals π (pi a constant) multiplied by the radius squared (r^2). Given a particular value for A or r, the equation can be solved (a value can be found) for the other variable. Given another equation simultaneously true, for example, $c = 2\pi r$, we can substitute $c/2\pi$ for r into the first equation. This gives a new equation, $A = c^2/4\pi$.

Algebra has been studied for many centuries. Babylonian and ancient Chinese and Egyptian mathematicians proposed and solved problems in words, that is, using "rhetorical algebra". However, it was not until the 3rd century that algebraic problems began to be considered in at form similar to those studied today.

Whereas many **Greeks** made decisive advances in geometry, they were known to produce only one algebraist, Diophantus of Alexandria (250 A.D.). Diophantus used an abridged notation for frequently occurring operations, and a special symbol for the unknown.

In the 3d century, the Greek mathematician *Diophantus* of Alexandria wrote his book *Arithmetica.* Of the 13 parts. originally written, only six still survive, but they provide. the earliest record of an attempt to use symbols to represent unknown quantities. *Dioplantus* did not consider general methods in *Arithmetica,* but instead solved a large number of practical problems.

Besides being the first to use symbols systematically in algebra, Diophantus was also the first to give general rules for the solution of an equation. An example, in modern notation,

$8x – 11 - 2x + 5 = x – 4 + 3x + 10$,

Rearranged in the form;
$8x + 5 + 4x + 3x + 10 + 11 + 2x$ or
$8x + 9 = 6x + 21$.

Diophantus then gives the following rule: "it will be necessary to subtract like from like on both sides, until one term is found equal to one term." What he meant was that one must subtract 6x from 8x and 9 from 12, so that there is only one term on each side.

Thus $8x - 6x = 21 - 9$

$$2x = 12$$
$$x=6$$

Diophantus also had methods for solving simultaneous and quadratic equations. He did not recognise negative numbers, so he had to distinguish three cases for quadratics:

1. $ax^2 + bx = c$

2. $ar^2 = bx + c$

3. $ax^2 + c = bx$

where a, b, c >0.
Each of these cases had its own method of solution, which correspond to the following expressions.

1. $X = \frac{\sqrt{\left(\frac{b}{2}\right)^2 + ac - \frac{b}{2}}}{a}$

2. $x = \frac{\sqrt{(\frac{b}{2})^2 + ac + b/2}}{a}$

3. $x = \frac{\sqrt{\left(\frac{b}{2}\right)^2 - ac + \frac{b}{2}}}{a}$

The fourth possible case $ax^2 + bx + c = 0$, (a, b, c > 0) does not occur in Diaphantus' work, since it never admits a positive solution.

Diaphantus′ name is today commemorated in the term "Diophantine equation" which is an equation for which only positive integer solutions are required. For example, $x^3 = 2 + y^2$, as a Diophantine equation, only has the solution
x=3, y=5.
Three of the problems that Diophantus worked on in one of his books titled **Arithmetica** include

1. Find two square numbers such that when one forms their product and adds either of the numbers to it, the result is a square number.

2 Find three numbers such that their sum is a square number and the sum of any two of them is a square number.
3. Find two numbers such that their sum is equal to the sum of their cubes.

By number, Diophantus means "positive rational number" Several Indian mathematicians carried out important work in the field of algebra in the 6th and 7th centuries. These include *Aryabhatta,* whose book entitled *Aryabhatta* included work on linear and quadratic equations, and

Brahmagupta, who presented a general solution for a quadratic equation.
The next major development in the history of algebra was the book *al-kitab al-muhtasar fi hisab wa'l-muqabala ("*compendium on calculation by completion and balancing), written by the Arabic mathematician Al- Kharza in the 9th century. This book developed method for solving six different types of quadratic equations, and contained the first systematic consideration of the subject. separately from number theory.

In about 1100, the Persian mathematician Omar Khayyam wrote a treatise on algebra based on Euclid's methods. In it he identified 25 types of equations and made formal distinction between arithmetic and algebra. During the 12th century, Al-Khwarizmi's works were translated and become available to Western scholars. In the 13th century *Leonardo Fibonacc*i wrote some important and influential books on algebra.

Rules for solving *cubic equation*s were discovered about 1515 by *Scipione del Ferro* (c.1465-1526), and for the quartic equation by Ludovico Ferrari (1522-1565) about 1545. In 1824 *Niels Henrik Abel* (1802-1829) finally proved that, in general, it is not possible to give general rules of this kind for solving equations of the fifth degree or higher.

Further contributions to the symbols used in algebra were made in the late 16th century and the 17th century by Francois Viete (1540-1603) and *René Descartes*, who introduced modern notations (for example the use of ') and also showed that problems occurring in geometry can be expressed and solved in terms of algebra. Complex and negative roots were late discoveries, and took some time to become accepted. In 1799, *Karl Friedrich Gauss* proved the fundamental theorem of algebra, which had been proposed as early as 1629.

The development of symbolic algebra went through three distinct stages namely:

a. Rhetorical algebra where equations are written in full sentences. For example, the modern equation x+5=7 was written in rhetorical form "Something plus five equals seven" or "Something plus 5 equals 7" Rhetorical algebra was first developed by the ancient Babylonians and remained dominant up to the 16th century.
b. Syncopated algebra where some symbolism is used which does not contain all of the characteristics of symbolic algebra. For instance, there may be a restriction that subtraction may be used only once within one side of an equation, which is not the case in symbolic algebra.

c. Symbolic algebra where full symbolism is used. Symbolic algebra was fully developed by Francois Viete (16th century).

Quadratic equations played an important role in early algebra; and throughout most of history. Until the early modern period, all quadratic equations were classified as belonging to one of three categories.

i. $x^2 + px = q$

ii. $x^2 = px + q$

iii. $x^2+q = py$, where p and q are positive.

This trichotomy comes because quadratic equations of the form 0, with p and q positive, have no positive roots. There were four conceptual stages in the development of algebra that occurred alongside the changes in expression. The four stages were as follows:

1. Geometric stage, where the concepts of algebra are largely geometric. For instance, an equation of the form $x^2= A$ was solved by finding the side of a square of area A.
2. Static equation-solving stage, where the objective is to find numbers satisfying certain relationships.
3. Dynamic function stage, where motion is an underlying idea. Algebra did not decisively move to dynamic function stage until Gottfried Leibniz.
4. Abstract stage, where mathematical structure plays a central role. Abstract algebra is largely a product of the 19th and 20th centuries.

Key ideas

Key I
• Properties of binary operations • Types of numbers • Every composite natural number can be expressed as a product of primes. • Highest Common Factor (HCF) or the Greatest Common Divisor (GCD) of two or more given natural numbers is the greatest number which is a factor of the given numbers

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of properties of binary operations in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of properties of binary operations equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

UNIT 4: INTRODUCTION TO PHILOSOPHY OF MATHEMATICS

If mathematics is regarded as a science, then the philosophy of mathematics can be regarded as a branch of the philosophy of science, next to disciplines such as the philosophy of physics and the philosophy of biology. However, because of its subject matter, the philosophy of mathematics occupies a special place in the philosophy of science. Whereas the natural sciences investigate entities that are located in space and time, it is not at all obvious that this is also the case for the objects that are studied in mathematics. In addition to that, the methods of investigation of mathematics differ markedly from the methods of investigation in the natural sciences. Whereas the latter acquire general knowledge using inductive methods, mathematical knowledge appears to be acquired in a different way: by deduction from basic principles. The status of mathematical knowledge also appears to differ from the status of knowledge in the natural sciences. The theories of the natural sciences appear to be less certain and more open to revision than mathematical theories. In this unit, we shall be looking at what philosophy is, Plato and Platonism, Formalism, Intuitionism, Absolutism, and end with Fallibilism.

Learning outcome(s)

By the end of the unit, you should be able to explain:

1. The philosophy of mathematics
2. Plato and Platonism;
3. Formalism, their tenets, and the implication for teaching and learning of mathematics;
4. Intuitionism, their tenets, and the implication for teaching and learning of mathematics
5. Absolutism, their tenets, and the implication for teaching and learning of mathematics
6. Fallibilism, their tenets, and the implication for teaching and learning of mathematics

SESSION 1: WHAT IS PHILOSOPHY OF MATHEMATICS?

In this session, we will focus on explaining philosophy of mathematics; as well as the philosophical assumptions in the teaching and learning of mathematics. It is hoped that learners would be able support in achieving the aims of philosophy of mathematics.

Learning outcomes

By the end of the session, the participant will be able to explain the:

1. philosophy of mathematics
2. aims of philosophy of mathematics

The science of numbers is mathematics. Arithmetic and geometry were the two traditional branches of mathematics that dealt with numbers and shapes, respectively. Even if modern mathematics is more complex and works with a larger range of objects, geometry and arithmetic remain of utmost importance. On the one hand, mathematical facts appear to be inevitable and compelling, but on the other, the origin of their "truthfulness" is still a mystery. The study of mathematics looks into this problem. The study of mathematics' most fundamental ideas and logical structure, with an eye toward the unification of all human knowledge, is known as foundations of mathematics. Number, shape, set, function, algorithm, mathematical axiom, mathematical definition, and mathematical proof are some of the most fundamental mathematical concepts. The abstract nature of mathematical concepts poses unusual and original philosophical problems.

Philosophies are seen as explanations that make an effort to create some type of order out of the inherent disorder of a collection of experiences. A philosophy is the description of a theory about the nature of something. It is a process of clarifying and organising experiences and values; it seeks relationships between things that are typically perceived as disparate and identifies significant differences between things that are typically regarded as the same. A philosophy is a product of time, thus it could become old or need to be modified in light of new experiences.

The area of mathematics known as philosophy of mathematics focuses on the philosophical underpinnings, premises, and consequences of mathematics. It tries to explain the nature and methods of mathematics and to comprehend the use of mathematics in everyday life. This topic is both extensive and distinctive among its philosophical counterparts due to the logical and structural structure of mathematics itself.

The key issues in metaphysics and epistemology are strongly tied to those in mathematics philosophy. It appears that mathematics studies intangible things. This prompts inquiries into the characteristics of mathematical entities and the methods by which we might learn about them. It becomes hard to imagine that mathematical objects might possibly be a part of the physical world after all. In a different sense, the philosophy of mathematics has shown that there is some potential for applying mathematical techniques to philosophical issues pertaining to mathematics.

Philosophy of mathematics is concerned with two main issues: the first is the meaning of everyday mathematical expressions, and the second is the question of whether abstract objects actually exist. The first is a simple interpretation problem: What is the best method to understand common mathematical statements and ideas like "5 is prime," "2+2 4," and "There are infinitely many prime numbers"? Therefore, developing a semantic theory for the mathematical language is the primary goal of mathematics philosophy. Semantics is concerned with the meanings of specific terms in everyday speech.

The interest in the meanings question was sparked by two things: 1) It is not at all clear what the correct response is, and 2) The many responses appear to have significant philosophical ramifications. More particular, various mathematical interpretations appear to yield various philosophical conceptions of the nature of reality. There are arithmetic sentences that seem to make simple statements about particular items. For instance, the phrase "5 is Odd" appears to be a straightforward subject-predicate sentence of the type "P is Q." "5 is odd" seems to make a clear and straightforward statement about the number 5. However, this is where the mystery lies. First of all, it's unclear what the number 5 should be. What sort of thing is a number, secondly? While realists believe there are such things as numbers, certain antirealist philosophers believed there are simply no such things as numbers (as well as other mathematical objects).

There are a variety of opinions about what a number is, even among realists. Some realists believe that numbers are mental objects, similar to thoughts that individuals have in their heads. Other realists contend that numbers are aspects of the physical reality that are independent of human thought. There is a third theory on the nature of numbers, known as Mathematical Platonism, which has gained increasing traction throughout the course of philosophy. This theory held that numbers are abstract objects that are neither physical nor metallic. One of the oldest and most contentious issues in philosophy is whether or not abstract objects actually exist. The idea that such entities exist dates back to Plato, and there has been significant opposition to the idea at least since Aristotle. More than 2,000 years have passed since the start of this ongoing debate.

A mathematical object is an abstract item that appears in both mathematics and mathematical philosophy. Numbers, permutations, partitions, matrices, sets, functions, and relations are examples of mathematical objects. The objects of algebra include groups, rings, and so on. Hexagons, points, lines, triangles, circles, spheres, polyhedra, topological spaces, etc. are examples of objects in geometry.

Mathematical objects are frequently quite abstract and removed from our normal sensory experiences. The existence and character of mathematical things thus pose unique philosophical problems. Is a square floor tile different from a geometrica square, for instance? So, where is the square-shaped object? Is it in our heads, on the floor, or some other place? Regarding sets, Do 52 cards make up a set, or is there something else involved?

Such problems were treated seriously by the Greek philosophers of antiquity. Indeed, they frequently used geometry and mathematics as a reference in their general philosophical talks. Aristotle examined and criticised Plato's assertion that mathematical objects, such as the Platonic forms, must be completely abstract and have a unique, immaterial nature. Aristotle claimed that the geometrical square is a crucial component of the square floor tile, but that it can only be comprehended by ignoring other unnecessary factors like the precise measurements, the type of material used for the tiling, etc. These issues undoubtedly lend themselves well to philosophical inquiry and discussion.

The Absolutist and Fallibilist interpretations of the nature of mathematics were distinguished by Lerman (1983) as two opposing viewpoints. Four different categories of concepts—multiplism, absolutism, relativism, and dynamism—were postulated by Grouws (1992) and Copes (1979). Platonism and Formalism are the two main theories of mathematics that shaped its beginnings. There are numerous schools of thought on philosophy of mathematics, which is now being investigated along a number of different paths by mathematicians, logical theorists, and philosophers of mathematics.

According to mathematical realism, mathematical objects exist apart from the mind. As a result, rather than creating mathematics, people find it. Thus, only one type of mathematics can be found; for instance, triangles are actual objects and not the products of the mind. The proponents of mathematical realism consider themselves as the finders of things that exist in nature. Kurt Gödel and Paul Erdös are two examples. According to Gödel, there is an objective mathematical world that may be experienced similarly to how the senses work. However, the continuum hypothesis conjecture may not be determinable just on the basis of some principles (e.g., for each two items, there is a collection of objects consisting of only those two objects).
According to Aristotelian realism, mathematics investigates symmetry, continuity, and other qualities that are literally realised in the physical world (or in any other world there might be). They contend that mathematical concepts like numbers can actually be realised physically rather than existing in a "abstract" universe. For instance, the relation between a bunch of parrots and the all-encompassing "being a parrot" that separates the bunch into so many parrots makes the number 4 apparent.

Key ideas

Key I

- Mathematics is the science of quantity
- The abstract nature of mathematical objects presents philosophical challenges that are unusual and unique.
- Philosophy is regarded as an explanation, which attempts to make some kind of sense out of the natural disorder of a set of experiences
- philosophy is a function of time and so may become outdated or have to be altered in the light of additional experiences
- Philosophy of Mathematics is the branch of mathematics that studies the philosophical assumptions, foundations, and implications of mathematics
- Philosophy of mathematics is concerned with problems that are closely related to central problems of metaphysics and epistemology

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of philosophy of mathematics in the teaching and learning of mathematics?
- How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?

Discussion

- How has this session equipped you to be a better classroom practitioner?
- How has your idea of philosophy of mathematics equipped you to be a better mathematics teacher?

- How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SECTION 2: PLATO AND PLATONISM

In this session, we will focus on explaining Plato and Platonism philosophical perspective of philosophy of mathematics; as well as core issues in Platonism. It is hoped that learners would be able apply the aims of Platonism philosophical perspective in mathematics education.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain Plato and Platonism philosophical perspective of mathematics
2. explain the core issues in Platonism
3. apply the aims of Platonism philosophical perspective in mathematics education

One of the most significant supporters of mathematics in ancient Greece was the philosopher Plato. In 387 BC, Aristotle founded his Academy in Athens, emphasising mathematics as a means of learning more about reality. Because of its abstract nature, he claimed that geometry holds the key to discovering the secrets of the cosmos and that it is "the first necessity in the teaching of philosophers." Let no one who is uninformed of geometry approach here, reads a famous inscription above the entrance to Plato's Academy.

Mathematics was taught as a subfield of philosophy in Plato's Academy. As the "manufacturer of mathematicians," Plato's Academy produced notable mathematicians throughout antiquity, including Eudoxus. The Platonic Solids are five regular symmetrical three-dimensional shapes identified by Plato that have come to be known as the foundation of the universe. They are the tetrahedron (built with four regular triangles and representing fire in Plato), the octahedron (composed of eight triangles and representing air), the icosahedron (composed of twenty triangles and representing water), the cube (composed of six squares and representing earth), and the icosa (made up of 12 pentagons, which Plato described as "the god used for arranging the conste lla tions on the whole heaven").

The kind of realism known as mathematical platonism contends that mathematical concepts are abstract, devoid of spatiotemporal or causal qualities, and eternal and unchanging. It is frequently asserted that most individuals interpret numbers in this way. The well-known Pythagoreans of ancient Greece, who held the notion that the cosmos was essentially formed by numbers, before and possibly had an influence on Plato's beliefs.

The existence of mathematical entities—and how do we know about them—in detail is a key issue in mathematical Platonism. Do the mathematical entities live in a realm that is entirely apart from our physical one? How can we enter this other dimension and learn the reality about the beings there? The Ultimate Ensemble, a theory that holds that any structures that exist mathematically also exist physically in their own world, is one solution that has been put forth. More people have been drawn to mathematical Platonism throughout the history of philosophy. Numbers are abstract, non-physical, and non-metal objects, according to Platonists. They perceive abstract objects, but neither the real world nor human minds contain them. In actuality, they don't even exist in space and time.

The Platonists include notable figures like Plato (1941) and Thom (1971). Platonists believe that there are true mathematical statements that accurately describe these abstract objects and that there are abstract objects that are completely non-spatiotemporal, non-physical, and non-mental. They all concur that non-spatiotemporality is the real distinguishing characteristic of an abstract object. In other words, although abstract objects are completely non-mental and cannot be found anywhere in the physical realm, they have existed and will continue to exist forever. One may see the number 5 as an example. This does not imply, however, that the number 5 is merely an abstract concept. Since the number 5 is an abstract object, it exists independently of humans and their thoughts, just like the moon and stars, yet it is nonphysical, unlike the moon and stars.

Abstract objects are completely non-causal and unchangeable in the eyes of Platonists. Due to their lack of spatial extension and material construction, abstract things are unable to form cause-and-effect interactions with other objects.

Platonists also contend that such objects are accurately described by mathematical theorems. For instance, the theory of arithmetic describes how this series of abstract objects looks using the positive integers 1, 2, 3, and so on. There are some fascinating details about this pattern. Euclid demonstrated that there are an unlimited amount of prime numbers among the positive integers about 2,000 years ago. Platonists assert that the sequence of positive integers is a subject of study in the same way as astronomers study the solar system. In addition to numbers, there are numerous additional types of mathematical objects. These are abstract things like functions, sets, vectors, circles, etc. Thus, according to Platonist definitions, mathematics is the study of the nature of diverse abstract mathematical forms.

One of the most prevalent ideologies among mathematicians is Platonism. The age of it exceeds two million years. The strongest argument in favour of platonism is said to have been established by a German named Gottlob Frege in the late 19th century, albeit he did not change how the argument was expressed. Kurt Gödel, an Austrian, and William Van Orman Quine, an American, both put out theories in the 20th century to try and explain how people could learn about abstract objects. The Platonist viewpoint itself was unaltered by them either.

In conclusion, Platonists believe that mathematical objects exist and that this fact is independent of our understanding of them. The items are real and distinct with distinct attributes, some of which are known and some of which are not. Every problem in mathematics has a solution (whether we can determine it or not). Due to the existence of something, the mathematician does not create anything new. Nothing is invented by a mathematician because it already exists. He can only learn. Based on these principles, the primary goal of mathematics education is to impart knowledge that is beneficial in and of itself—good because it develops the mind rather than because it is useful in day-to-day life.

Key ideas

Key I
• Plato perceived mathematics as a way of understanding more about reality • Plato's Academy taught mathematics as a branch of philosophy • Mathematical Platonism is the form of realism that suggests that mathematical entities are abstract, have no spatiotemporal or causal properties, and are eternal and unchanging • Platonism is basically about where and how do the mathematical entities exist, and how do we know about them? • Platonists thus defined mathematics as the study of the nature of various mathematical structures, which are abstract in nature.

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of Platonism in philosophy of mathematics in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of Platonism in philosophy of mathematics equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SECTION 3: FORMALISM

In this session, we will focus on explaining formalism philosophical perspective of philosophy of mathematics; as well as core issues in formalism. It is hoped that learners would be able apply the aims of formalism philosophical perspective in mathematics education.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain the formalists philosophical perspective of mathematics
2. explain the core issues in formalism
3. apply the aims of the formalist’s philosophical perspective in mathematics education

The discovery that all of mathematics may be reduced to formal theories in the 20th century caused a major uproar. The emergence of the extreme philosophical philosophy known as formalism was one manifestation of this elation. Formalism holds that mathematics is merely a formal game that is primarily interested in the algorithmic manipulation of symbols. The symbols are nothing more than marks on paper or bits and bytes in a computer's memory. As a result, mathematics cannot assert to possess any kind of object-specific knowledge. Indeed, there is no such thing as a mathematical object.

In the arts, literature, or philosophy, formalism is the stress on form over meaning. According to formalists, the literal content produced by a practitioner is the only transcendent meaning associated with that discipline. Formalists are solely focused on "the rules of the game," and they hold the view that there is no other external truth that can be attained in addition to those predetermined guidelines.

Formalists contend that mathematics is nothing more than the symbols that the mathematician writes down, which are based solely on logic and a few simple rules. Disciplines based on axiomatic systems benefit from formalism. They think that axioms, definitions, and theorems make up mathematics. In mathematics, formal symbolic systems are of interest. Mathematicians are seen by formalists as a collection of these kinds of abstract advances, in which the concepts are merely symbols. Mathematics merely comprises perfect symbolic parts and lacks any tangible content. As a result, teachers of rules and formulas that encourage instrumental learning are born.

A particular school of thought in the philosophy of mathematics known as formalism emphasises axiomatic proofs via theorems. They regarded the study of formal axiom systems as mathematics. According to the formalist idea, mathematical and logical claims can be viewed as statements about the effects of particular string manipulation rules. A game called Euclidian geometry is played by moving around a collection of symbol strings called axioms in accordance with a set of guidelines known as the rules of inference to create new strings. Because the string that embodies the Pythagorean Theorem can be created by following only the given rules, this game can be used to demonstrate the validity of the Pythagorean Theorem. According to formalism, the truths revealed by logic and mathematics have nothing to do with sets, numbers, triangles, or any other specific subject. They actually don't have anything to do with anything. Unless they are given a purpose, their shapes and positions are meaningless.

David Hilbert, who made the initial effort to axiomatize all of mathematics, was a significant early proponent of formalism. He was attempting to demonstrate the consistency of number theory because he believed that there was some purpose and truth in mathematics. There must be some validity to number theory if it turns out to be consistent. Mathematics is separated from its semantic meaning in the eyes of strict formalists. They see arithmetic as simply the manipulation of symbols in accordance with predefined rules. They then make an effort to demonstrate the consistency of this system of rules, much like Hilbert did. Other formalists, such Haskell Curry, Alfred Tarksi, and Rudolf Carnap, saw mathematics as the study of formal axiom systems.

The second of Göde's incompleteness theorems, which implies that sufficiently expressive consistent axiom systems can never prove their own consistency, severely weakened Hilbert's attempts to construct a mathematical system that is both complete and consistent. You cannot demonstrate consistency in any axiomatic system that is sufficiently rich to contain classical arithmetic, according to Gödel's incompleteness theorem. It is impossible to demonstrate the consistency of this language by itself, hence in one case you must only utilise the formal language used to formalise this axiomatic system. Hilbert's attempt to fully codify all of number theory was severely thwarted by Gödel's work. However, Gödel did not believe that he completely refuted Hilbert's formalist viewpoint. The main difference is that, contrary to what Hilbert had intended, the proof theory could not be utilised to demonstrate the consistency of the entire field of number theory. The major argument against formalism is that it is too disconnected from the real mathematical concepts that keep mathematicians up at night.

One type of formalism is deductivism. The Pythagorean Theorem is a relative fact according to deductivism rather than an unalterable one. This means that you must accept the theorem, or rather, the interpretation of the theory you have given it, as a true statement if you interpret the strings in a way that makes the game's rules true.

Deductivism holds that all other propositions of formal logic and mathematics are also true. Formalism need not equate these logical sciences to useless games of symbols. Usually, the assumption is made that there is some interpretation in which the game's rules are valid. In response to formalism, a number of alternative doctrines have been promoted. One of them is constructivism, which holds that a sequence of purely mental constructions can be used to acquire mathematical knowledge. According to this perspective, mathematical objects only exist in a mathematician's head, making mathematical knowledge completely certain. However, the relevance of mathematics to the outside world is called into question.

Key ideas

Key I
• Formalism believed that mathematics is only a formal game, concerned solely with algorithmic manipulation of symbols. • Formalism is a certain school of thought in the philosophy of mathematics, which stresses axiomatic proofs through theorems • Formalism lends itself well to disciplines based upon axiomatic systems • Formalists regard mathematics as a collection of such abstract developments, in which the terms are mere symbols • Gödel's incompleteness theorem means that you cannot prove consistency within any axiomatic system rich enough to include classical arithmetic • Deductivism is one version of formalism

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of formalism in philosophy of mathematics in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of formalism in philosophy of mathematics equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SECTION 4: INTUITIONISM

In this session, we will focus on explaining intuitionism philosophical perspective of philosophy of mathematics; as well as core issues in intuitionism. It is hoped that learners would be able apply the aims of intuitionism philosophical perspective in mathematics education.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain the intuitionists philosophical perspective of mathematics
2. explain the core issues in intuitionism
3. apply the aims of the intuitionist’s philosophical perspective in mathematics education

L.E.J. Brouwer, a Dutch mathematician, developed the intuitive approach to mathematics (1881-1966). The philosophy is predicated on the notion that mathematics is a mental construct. Mathematical statements can only be shown to be true mentally, and communication among mathematicians simply serves to facilitate the development of the same mental process in various brains.

Intuitionism has a key component with the majority of other constructivist ideologies, which are typically logical and useful mathematical objects. Theoretically, one can derive algorithms that compute the elements and simulate the constructions whose existence is established in the proof from constructive proofs. The majority of constructivist theories are compatible with classical mathematics since they generally follow a stricter interpretation of the permitted constructions and connectives and quantifiers while making no extra assumptions. Almost all constructive societies subscribe to the same logic, known as intuitionistic logic.

This perspective on mathematics has broad ramifications for mathematical work on a daily basis. It is crucial to understand how time affects intuitionism because over time, propositions may become demonstrably true and, as a result, may start to qualify as valid according to intuitionism even if they weren't earlier. In addition to developing intuitionism as a philosophy, Brouwer applied these ideas to mathematics, particularly the theory of sets and the theory of the continuum. Even though they held opposing opinions on the subject, some of the most renowned mathematicians of his time accepted his philosophy as a genuine alternative to classical logic, despite the fact that many others found it awkward. One of them, for instance, was Kurt Gödel.

David Hilbert and Brouwer had a disagreement that rocked the mathematical community at the start of the 20th century and was brought on by the emergence of mathematical paradoxes. Philosophers are compelled to admit that mathematics lacks an ontological and epistemological foundation.

According to Brouwer, mathematics is a mind-created, languageless concept. The only a priori concept is time. He separates two types of intuition: The primary concern of the first act is the total dissociation of mathematics from mathematical language and, by extension, from linguistic phenomena covered by theoretical logic. He understood that intuitionistic mathematics is primarily a languageless mental activity with roots in the perception of temporal movement. The natural numbers are created by intuitionism's first act, but it also indicates that the rules of reasoning are severely constrained.

The second act deals with allowing two methods for creating new mathematical entities: first, in the form of more or less freely proceeding infinite sequences of previously acquired mathematical entities; and second, in the form of mathematical species, which are properties supposable for previously acquired mathematical entities that satisfy the requirement that if they hold for a particular mathematical entity, they also hold for all mathematical entities which have been defined.

The two intuitionistic acts serve as the cornerstone of Brouwer's philosophy. These fundamental ideas lead to the conclusion that intuitionism is distinct from Platonism and Formalism. This is due to the fact that it neither holds that mathematics is a game of symbols played by predetermined rules nor does it postulate an external mathematical reality. According to Brouwer, mathematics is communicated through language, but the latter is independent of the former. The freedom that the second act offers in the formation of infinite sequences is what sets intuitionism apart from other constructive viewpoints on mathematics that maintain that mathematical objects and arguments should be computable.

As a result, Brouwer's intuitionism is neither Platonism nor Formalism because it is grounded in a knowledge of time and a belief that mathematics is an invention of the free mind. It is a kind of constructivism, but only in the broadest sense, as many constructivists do not subscribe to all of Brouwer's tenets.

According to mathematics intuitionism, "there are no non-experienced mathematical truths." According to intuitionists, the corrigible part of mathematics has to be rebuilt using Kantian notions of being, becoming, intuition, and knowing. According to Brouwer, the a priori forms of the volitions that guide how we perceive empirical objects give rise to mathematical objects. He disregarded the law of excluded middle, proofs by contradiction, and the axiom of choice as being useful for mathematics of any kind. However, the lack of a precise definition of the word "explicit construction" in intuitionism has drawn criticism.

Key ideas

Key I

- Intuitionism is based on the idea that mathematics is a creation of the mind
- Intuitionism shares a core part with most other forms of constructivism which is generally Constructive mathematical objects and reasoning
- Brouwer described mathematics as a languageless creation of the mind
- Truth of a mathematical statement can only be conceived via a mental construction that proves it to be true and the communication between mathematicians only serves as a means to create the same mental process in different minds
- Brouwer held that mathematical objects arise from the a priori forms of the volitions that inform the perception of empirical objects

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of intuitionism in philosophy of mathematics in the teaching and learning of mathematics?

• How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of intuitionism in philosophy of mathematics equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SECTION 5: ABSOLUTISM

In this session, we will focus on explaining absolutism philosophical perspective of philosophy of mathematics; as well as core issues in absolutism. It is hoped that learners would be able apply the aims of absolutism philosophical perspective in mathematics education.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain absolutism philosophical perspective of mathematics
2. explain the core issues in absolutism
3. apply the aims of absolutism philosophical perspective in mathematics education

According to absolutists, mathematics is a body of knowledge whose truths are universally acknowledged to be necessary and unquestionable. The foundation of the entire system is based on a set of self-evident presumptions. Since mathematics governs nature and helps to develop the universe through its immutable consistency over time and space, they view it as almost being independent of humanity. One of mathematics' most potent charms is consistency. The straightforward experiment of repeatedly dropping an apple and timing how long it takes to hit the ground serves as one kind of proof. Never has anyone objected to this. The nature of mathematics was broadly accepted for several thousand years.

All of our understanding of mathematics is based on axioms. Logic was used to construct the theorems and proofs that made up the subject itself from axioms. Because the laws of nature were independent of human existence, mathematics was often considered as being value-free. Their worldview prohibited challenging fundamental mathematical concepts or the methods used to generate them. Absolutists are a group of educational ideas that include the industrial trainer and ancient humanists. The student is seen as an empty vessel, and the teacher as an authoritarian.

The absolutists believed that mathematics is a body of knowledge that is objective, absolute, certain, and incorrigible and that is based on the solid principles of deductive reasoning. Mathematical absolutist philosophies focus on the epistemic task of supplying mathematical knowledge without exception rather than on descriptive philosophies. They attribute mathematics to the use of strict logical structures that were created for epistemological reasons. They contend that mathematical knowledge is: 1) timeless, even though new theories and truths may be added; 2) superhuman and ahistorical, since the history of mathematics has no bearing on the nature or justification of mathematical knowledge; 3) pure isolated knowledge, which just so happens to be useful due to its universal validity; and 4) value-free and culture-free, also for the same reason.

According to absolutism, mathematics is seen as being rigid, fixed, logical, absolute, inhuman, chilly, objective, pure, abstract, distant, and overly rational. If teachers have this perspective on mathematics, then it is likely that they will convey this perspective to the pupils they teach. Due to their influence, students are frequently assigned unrelated routine arithmetic assignments in school and expected to use previously learned techniques. Such instructors emphasise that each mathematical problem has a specific, fixed, and indisputable correct solution, and they will not tolerate any inability to arrive at this solution.

These philosophies are presented from the perspective of a mathematician. However, the way mathematics is taught in schools and how it is taught has an impact. When it comes to educational practises, different mathematical ideas provide wildly divergent results. The absolutist view of mathematics as being cold, absolute, and inhuman is confirmed by several classroom experiences from both teachers and students. This view is frequently linked to unfavourable attitudes toward mathematics.

Key ideas

Key I

- Absolutists regarded mathematics as an objective, absolute, certain, and incorrigible body of knowledge, which rests on the firm foundations of deductive logic
- Absolutists hold the view that mathematics is a body of knowledge whose truths appear to everyone to be necessary and certain
- Absolutists see mathematics as almost independent of humankind, existing as it does in its government of nature, which builds the universe together with its unfailing consistency across time and space
- Axioms provided the basis for all mathematical knowledge
- Mathematics was widely regarded as value-free, at least partly because the laws of nature were not dependent upon the presence of humankind
- Theorems and proofs which constituted the subject itself were built on axioms using logic

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of absolutism in philosophy of mathematics in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of absolutism in philosophy of mathematics equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SECTION 6: FALLIBILISM

In this session, we will focus on explaining fallibilism philosophical perspective of philosophy of mathematics; as well as core issues in fallibilism. It is hoped that learners would be able apply the aims of fallibilism philosophical perspective in mathematics education.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain fallibilism philosophical perspective of mathematics
2. explain the core issues in fallibilism
3. apply the aims of fallibilism philosophical perspective in mathematics education

Part of fallibilism's emergence was a response to absolutism. in general The Fallibilists countered that the Absolutists' belief in an infallible body of reasoning that is acknowledged by everyone is a big lie. The awareness that there are no dependable sources of information and that all sources are faulty leads to fallibilism, a school of thought in and of itself. According to them, mathematics is fundamentally a human endeavour that was created by humans, making it vulnerable to human error. In contrast to absolutists, they do not adhere to a set of beliefs based on unquestionable truths. Fallibilists contend that we should openly acknowledge mathematics' limitations.

They put out the idea that mathematics is a human endeavour that is fallible, historical, and dynamic. According to the Fallibilist, social processes produce mathematics. Mathematical knowledge must always be subject to change, including both its conceptions and its proofs. This viewpoint accepts the methods used by mathematicians, their history and applications, the significance of mathematics in human society, as well as ethical and educational considerations. They disagree with the idea that there is a single, fixed, and indelible hierarchical structure, but they do not dismiss the importance of logic and organisation in mathematics. They acknowledge that mathematics is composed of numerous overlapping structures that have grown, disintegrated, and then regrown throughout history, much like trees do in a forest.

According to fallibilism, mathematics is connected to a variety of social practises, each with its own history, people, institutions, and social locations, as well as symbolic forms, purposes, and power dynamics. Fallibilism challenges the absolutist view of mathematics as a body of pure and perfect abstract knowledge existing in a superhuman, objective realm. Some examples of such activities are academic research, classroom mathematics, and ethnomathematics.

Fallibilists contend that while mathematical knowledge is a contingent social construction, it is fixed and should be transmitted to learners in this way as long as it is still accepted by the mathematical community. They also contend that questions of school mathematics are only uniquely definable as right or wrong with reference to its conventional Corpus of knowledge. Fallibilists hold that there can be no absolute assurance in knowledge, or at the very least, that any claim to knowledge could theoretically be false. They acknowledge that empirical knowledge may constantly be changed through additional observation, meaning that any such understanding could end up being incorrect. Although Socrates and Plato are thought to have known about fallibilism, the formal idea is most closely connected with the late 19th-century philosopher Charles Sanders Peirce and others like W.V.O. Quine and Karl Popper (1902 1994). It affected the growth of C. S. Peirce's, William James', and John Dewey's pragmatism.

Key ideas

Key I

- Fallibilists argued that the Absolutists held view of a universally accepted, infallible body of reasoning is a grand illusion
- Fallibilists regard mathematics as an essentially human pursuit, invented by humans, and this makes it prey to human fallibility
- Fallibilists positioned the image of mathematics as human, corrigible, historical and changing
- Mathematics to the Fallibilist is the outcome of social processes
- Mathematical knowledge is to be eternally open to revision, both in terms of its proofs and its concepts
- Fallibilists believe that absolute certainty about knowledge is impossible, or at least that all claims to knowledge could, in principle, be mistaken.

Reflection

- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of fallibilism in philosophy of mathematics in the teaching and learning of mathematics?

• How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of fallibilism in philosophy of mathematics equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

UNIT 5: PROOFS IN MATHEMATICS

The definition of a proof is the logical way in which mathematicians demonstrate that a statement is true. In general, these statements are known as *theorems* and *lemmas*. A theorem is a declaration that can be determined to be true using mathematical operations and arguments. On the other hand, a lemma is like a smaller theorem that is used to prove a much greater theorem is true.

Learning outcomes

By the end of the unit, participants should be able to explain:

1. proofs in mathematics with specific examples;
2. inductive reasoning and give some examples;
3. proof by Mathematical Induction and perform examples of proof by mathematical induction
4. Deductive Reasoning and give mathematical examples

SESSION 1: DEFINITION OF PROOF

In this session, we will focus on explaining proofs; as well as mathematical proofs. It is hoped that learners would be able apply mathematical proofs in learning mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain proofs
2. explain mathematical proofs
3. apply mathematical proofs in teaching and learning

Our mathematical calculations frequently contain errors due to inaccurate measurement or a misunderstanding of the formulas we employ. In mathematics, we frequently want to be certain that our actions are correct, which is one reason why we require proofs. Simply testing our logic is proof. The Latin verb probare, which means "to test," is where the term "proof" originates.

One of the most significant contributions of ancient Greek mathematics is the creation of mathematical proof. Euclid (300 BCE), who established the axiomatic technique, which is still in use today, revolutionised proofs by beginning with undefined concepts and axioms. This desire of certainty was fostered by the Greeks. They viewed mathematics as a means of creating a world of absolute truth, layering one fact upon another to ensure its accuracy. They first based a significant portion of their geometrical thinking on the presumption that any two lines could be compared by identifying any unit small enough that their lengths were whole-number multiples of that unit; all lines were therefore considered to be "commensurable." However, it was found that the square root of two was irrational since the diagonal of a unit square was incommensurable with the side of the square. As a result, they were compelled to reconsider their arguments and were able to reconstruct their geometry. Mathematicians were reminded by this instance of the value of thoroughly proving everything.

Mathematical propositions must first be proved (or refuted) in order for us to know if they are true or wrong. Presenting a sound mathematical proof is the only way to confidently establish the veracity of a conjecture.

A proof is described as a series of statements, each of which is either correctly deduced from the claims that came before it, according to the Harper Collins Dictionary of Mathematics. In a similar vein, The Penguin Dictionary of Mathematics describes a proof as a line of reasoning based on principles of inference that leads to a conclusion. In our formal language, proofs are nothing more than the manipulation of symbols beginning with a set of precepts we refer to as axioms.

A logical argument that confirms something's truth without a shadow of a doubt is referred to as a proof. It is the result of deriving one idea from another. The methods or procedures used to establish the veracity of a proposition are referred to as a proof. It is a justification that persuades other mathematicians that a claim is accurate. A strong argument for the truth also aids in their comprehension. A proof is a rigorous argument that is presented in clear language with the goal of persuading the listener that a statement is true. In the discipline of mathematical logic, the idea of a proof is codified. A formal proof is one that is written in a formal language as opposed to a natural language and is described as a series of formulas where each formula follows logically from the formulas before it.

A mathematical statement's proof is a deductive argument. Theorems and other previously proven statements can be incorporated into the argument. A proof can, in theory, be traced back to axioms, which are self-evident or presumptive truths. In contrast to inductive or empirical arguments, proofs are instances of deductive reasoning.

To prove anything is to persuadely demonstrate its truth or validity through an argument. A proof is strictly a series of facts that are inferred from axioms or previously known facts. It is implicitly assumed that a conclusion that meets the principles of logic is strong enough to support it. But occasionally, a proof may contain an error unintentionally. The evidence may then provide a strong case for the correctness of a fact, which could be true or untrue in and of itself. A proof is considered a fallacy if it offers a strong case for the truth of a false claim.

A mathematical proof is a series of inferences that show a set of axioms to be true. Mathematical proof and common reasoning share certain similarities. A mathematical proof is an inferential argument where other previously established claims, such as theorems, are utilised to support the claim being made.

In mathematics, we can demonstrate that our actions are wholly correct. This is so that mathematics can be based solely on reason rather than on imperfectly understood physical principles or unpredictable human behaviour. In mathematics, unlike the real world, the rules are established by us, allowing us to know everything we require to be assured of what will occur. For instance, we can clarify what addition is before demonstrating that adding (b+a) always produces the same outcome as adding (a+b).

Things must be supported by evidence if we are to avoid being duped. The mere fact that something holds true every time we try it does not guarantee that it will do so in the future. For example, we can be confident that a+b=b+a because we understand how addition works and know that this rule is a natural outcome of how addition works, not because we have always observed it to work that way.

To refute a statement is to demonstrate its falsity, possibly by an illustration or by demonstrating its negation. Finding a counter example, or an example that meets the requirements of the statement's premises but not its conclusion, is frequently all that is necessary to refute it.

Proofs in mathematics provide a lot of light. Possessing a solid proof on paper might be a sign that you have a solid grasp of the issue. Sometimes, the efforts to support a supposition necessitate a greater comprehension of the underlying theory. Even if attempts are made to establish the speculation but fail, one learns a great deal and learns a lot. The mathematical elegance of a proof can be valued as an aesthetic object.
We often use two types of thinking in daily life: inductive reasoning and deductive reasoning.

Key ideas

Key I

- In mathematics, we usually want to be sure that what we do is right and this is one reason why we need proofs
- Proof is a logical argument that establishes, beyond any doubt, that something, is true
- Proofs are simply the manipulation of symbols in our formal language starting from certain rules that we call axioms
- Mathematics defines a proof as a chain of reasoning using rules of inference, ultimately based on a set of axioms that lead to a conclusion
- Proof is a deductive argument for a mathematical statement
- Mathematical proof is a process of deductions using axioms to demonstrate that a process is true. Mathematical proof has something in common with everyday argument

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of mathematical proofs in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of mathematical proofs equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 2: INDUCTIVE REASONING

In this session, we will focus on explaining inductive reasoning; as well do examples of inductive reasoning. It is hoped that learners would be able to apply inductive reasoning in learning mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain inductive reasoning
2. do examples of inductive reasoning

Drawing a broad inference from what is happening around us is an example of inductive reasoning. To draw a conclusion, one must use experience, sensory impressions, and observations; in other words, one must predict what will happen in the near future based on observations from the past. For instance, I'm about to eat some beans and I've always gotten gas after eating beans. Inductive reasoning involves projecting patterns, regularities, and resemblances that have already been detected onto future experiences in order to make judgements about the unobserved based on witnessed examples.

Is 414, 612, 522, 1602, 7020, 1121121, and 100314 divisible by 9? Why or why not, please.
Analyzing the 9 times table critically reveals a pattern. In all of the products, the digit sum is 9. Will the nine times table apply in all situations?
A mathematician would frequently attempt to tackle an easier, but related problem first when faced with a challenging one. It is frequently the case that finding symmetry and pattern is the most elegant and efficient way to solve a mathematical problem. Let's think about the following issues:

Example 1
Without using calculator or tables, compute 10 + 123456789 × 9.

Solution:
Study the pattern: $2 + 1 \times 9 = 11$
$3 + 12 \times 9 = 111$
$4 + 123 \times 9 = 1111$

Any pattern observed in the answer as well as in the sequence of problems? Now predict the answer to the next problem?

$$5 + 1234 \times 9 = \cdots$$

Continue the computation like this until you see a pattern and then predict the answer to

$$10 + 123456789 \times 9 = \cdots$$

Example 2
Study the following pattern

$9 \times 1 - 1 = 8$
$9 \times 21 - 1 = 188$
$9 \times 321 - 1 = 2{,}888$

(a) Write down the next problem?

(b) What is the next answer in the sequence?
(c) Use the patterns to predict the answer to

$$9 \times 987654321 - 1$$

Solution
The next problem and the answer are **9 × 4321 -1 = 38,888**
Using the patterns, we predict answer to the problem as:
9 × 987654321 - 1 = 8,888,888,588

Inductive reasoning is a type of reasoning that uses specific data to infer a more general conclusion that is thought to be likely but yet leaving room for the possibility that the conclusion might not be accurate. A crucial way of thinking, frequently referred to as the scientific method, is inductive reasoning. For instance, inductive reasoning is crucial for scientific discovery because it is employed for developing ideas, hypotheses, and relationships. When conducting tests to find diverse natural laws, scientists employ inductive reasoning.
Inductive reasoning is a technique used by statisticians when drawing conclusions from data. It entails deducing a general conjecture from specific facts or isolated occurrences. In other words, a generalisation is drawn based on a small number of observed events. The more distinct events we notice, the better able we are to generalise in the right way. A tentative inductive conclusion may need to be amended in light of fresh information. There might be a counter

example, but continuous inductive reasoning should be applied. There is no logical progression from premises to conclusion in inductive reasoning. The presumptions provide solid justification for accepting the result. Inductive arguments typically start with premises that are supported by data or observations. But since there isn't always a logical connection between the premises and conclusion, it is always possible for the premises to be accurate while the conclusion is incorrect.

Sometimes using patterns can help answer difficult math problems. For instance, what if we were tasked with counting the squares on a 9 × 9 card?
Start with a 1 by 1 Small Square, then move on to a 2 by 2 square, a 3 by 3 square, and so on. As you go, look for a pattern.

Fibonacci numbers
Fibonacci numbers are defined by the recurrence relation:
f_1 =1, $f_{2\,=}$ 1 and for n$>$ 2, $f_n = f_{n-1} + f_{n-2}$.
So, the first few Fibonacci Numbers are:
1, 1, 2, 3, 5, 8, 13, 21.34, 55, 89…
Now, write down the next four numbers in the sequence.
Are your numbers same as these 144, 233, 377, 610?
There are numerous curious properties of the Fibonacci Numbers. Once guessed, most such properties can be verified by induction. Here are a few of the properties.

(i) For every n ≥ 1, f $\epsilon\, n$ is even.
(ii) For every 121, 14 is divisible by 3. Try them yourself
Although mathematicians often proceed by inductive reasoning to formulate new ideas, they are not content to stop at the "probable stage. It is simply based upon the principle of the uniformity of nature. We can only evaluate the reasonability of inductive arguments as strong, moderate, or weak arguments and not as cogent. An inductive argument is valid, only if
a) the projection from premise to conclusion is correct
b) the sample represented in the premise is large enough (representative enough).
If you use inductive reasoning, you have to be open to revising your conclusion when new evidence comes to light, and that's what scientists generally do. They often formalize their predictions into theorems and then try to prove those theorems deductively.

Key ideas

Key I

- Inductive reasoning involves drawing a general conclusion from what we see around us
- Inductive reasoning is reasoning from experience, sense perceptions, and observations to form a conclusion
- forming an expectation of what will happen in the near future, based upon past observations

Reflection

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- What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts of inductive reasoning in the teaching and learning of mathematics?
- How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?

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Discussion

- How has this session equipped you to be a better classroom practitioner?
- How has your idea of inductive reasoning equipped you to be a better mathematics teacher?
- How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

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SESSION 3: PROOF BY MATHEMATICAL INDUCTION

In this session, we will focus on explaining proof by mathematical induction; as well do examples of proof by mathematical induction. It is hoped that learners would be able to apply proof by mathematical induction in learning mathematics.

Learning outcomes
By the end of the session, the participant will be able to:

1. explain proof by mathematical induction
2. do examples of proof by mathematical induction

Mathematical induction is a method of deduction, not a form of inductive reasoning. In proof by mathematical induction, a single "base case" is proved, and an "induction rule" is proved that establishes that any arbitrary case implies the next case. This avoids having to prove each case individually. The idea of mathematical induction is that a finite number of steps may be needed to prove an infinite number of statements $P_{1,}\ \ P_{2,}\ P_{3\ldots}$ *Mathematical Induction* is an important tool in Mathematics. It is a way of proving mathematics statements deductively for all positive integers.

The complete proof for the infinity of cases is made to depend upon just two steps.

1. The usually easy task of checking the formula in the case n = 1 (or sometimes the first positive integer for which the formula has meaning is the integer that is tested here). This may be called the Basis for the Induction; and

2. The making of the hypothesis, H, that the formula is correct in all cases 1, 2, 3,…n and then proving that, as a consequence of the hypothesis, H, and previously known theorems, the formula is correct in the case (n+1). This step is called the *Core of the Induction Proof*.

Finally, if 1 and 2 have been established, we can apply the axiom of the mathematical induction and make the conclusion that the formula under consideration is true for all positive integers (or positive integers beginning with the smallest integer that can be used in step 1)

The process of *Principle of Mathematical Induction* implies that
(i) Show it is true for n=1
(ii) Assume it is true for n = k
(iii) Show it is true for n = k+1
Conclusion. The Statement is true for all n$\geq$ 1

The key word in step (ii) is assume. You are not trying to prove it's true for n= k, you're going to accept in faith that it is, and show it is true for the next number, n= k+1. The assumption is known as *induction hypothesis*. If it later turns out that you get a contradiction, then the assumption was wrong.

It may be helpful to think of the domino effect illustrated in the diagram shown. An infinite number of dominoes are arranged in succession as shown.

- The first domino will fall.
- Whenever a domino falls, its next neighbor will also fall.
- So it is concluded that all of the dominoes will fall. and that this fact is inevitable.

DIAGRAM…..

That is, to prove that a property known to hold for one number holds for all natural numbers: Let N = (1, 2, 3, 4,…) be the set of natural numbers, and P(n) be a mathematical statement involving the natural number belonging to N such that
(i) P(1) is true, i.e., P(n) is true for n=1
(ii) P(n+1) is true whenever P(n) is true, i.e., P(n) is true implies that P(n+1) is true. Then P(n) is true for all natural numbers n.

Example
Prove that for all the integers n, the number 3^{2n} -1 divisible by 8.
Let P_r : 3^{2n} -1 is divisible by 8.

Proof:

Basis: If n = 1, then 3^{2n} -1 =$3^{2(1)} - 1 = 9 - 1 = 8$ is divisible by 8. P_1 is true.
Core: Assume $3^{2k} - 1$ is divisible by 8
For some integer x, we have

$$3^{2k} - 1 = 8x$$

$$3^{2k} = 1 + 8x$$

$$3^2 \times 3^{2k} = 3^2(1 + 8x)\ (Multiplying\ by\ 3^2)$$

$$3^{2k+2} = 9(1 + 8x) = 9(8x) = 1 + 8 = 9(8x)$$
$$3^{2k+2} = 1 + 8(1 + 9x)$$
$$(Factor\ out\ 8\ or\ apply\ the\ distributive\ property)$$
$$3^{2k+2} - 1 = 1 + 8(1 + 9x)$$

This implies that $3^{2(k+1)} - 1$ is divisible by 8. Since P_1 is true and P_{k+1} is true whenever P_k is true, P_r is true for all positive integers.

Key ideas

Key I
• Mathematical induction is a method of deduction, not a form of inductive reasoning • In proof by mathematical induction, a single "base case" is proved, and an "induction rule" is proved that establishes that any arbitrary case implies the next case • we can apply the axiom of the mathematical induction and make the conclusion that the formula under consideration is true for all positive integers (or positive integers beginning with the smallest integer that can be used in step 1)

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts proof by mathematical induction in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
Discussion • How has this session equipped you to be a better classroom practitioner? • How has your idea of proof by mathematical induction equipped you to be a better mathematics teacher? • How can you contribute to achieving the aims and vision of the BSC (standards-based curriculum)?

SESSION 4: DEDUCTIVE REASONING

In this session, we will focus on explaining deductive reasoning; as well do examples of inductive reasoning. It is hoped that learners would be able to apply deductive reasoning in learning mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

1. explain deductive reasoning
2. do examples of deductive reasoning

When using deductive reasoning, you begin with a generalisation you are confident is true and come to conclusions about a particular situation. For instance, if you are aware that all sheep enjoy eating grass and that the animal you are currently looking at is a sheep, you may be certain that it does the same. Only if your premise is incorrect—that is, if you are wrong that all sheep like grass—or if the object in front of you is not a sheep—can this line of reasoning go awry. But if those two assumptions are true, then your conclusion—that it is true everywhere and forever—follows logically from your premises. Pythagoras' theorem is just one example of a statement that mathematics tries to prove is true forever and everywhere. Deductive reasoning is the foundation of mathematics for this reason.

A mathematical proof is an argument that draws conclusions about the assertion being proved from other conclusions that are unquestionably true. For instance, if you know the values of the first two angles in a triangle, you can infer the third angle's value from the fact that all triangles drawn in a plane have angles that sum up to 180 degrees.

Axioms, definitions, and prior theorems are logically combined to establish the conclusion via direct proof. To demonstrate, for instance, that the sum of two even numbers is always even, consider the following:

Let the two even integers be $x \, and \, y$. They can thus be written as $x = 2a \, and \, y = 2b$, respectively, for integers $a \, and \, b$. Then the sum $x + y = 2a + 2b = 2(a + b)$.
Therefore, $(x + y)$ has 2 as a factor and, by definition, is even.
Hence the sum of any two even integers is even.
This proof uses the definition of even integers, the integer properties of closure under addition and multiplication and distributivity. In algebraic proof we show that a result is true for x, and providing no arithmetic rules have been broken, it is true for any number subject the original boundaries set on x, for example, it must be a positive whole number.

Example 1

The n^{th} term of the sequence of triangular numbers 1, 3, 6, 10, 15… is given by $\frac{n}{2}(n + 1)$. Prove that eight times any triangular number is one less than a square number.

Solution

If it is a triangular number, then we need to prove that $8T + 1$ is a square number, where T is a triangular number. $8T + 1$ is given by $4n(n + 1)$,
which simplifies to $4n^2 + 4n + 1$.
But $4n^2 + 4n + 1 = (2n + 1)^2$ which is a square number- we have proved the result.

Example 2: The n^{th} term of the so- called rectangular numbers is n(n+1). Prove that rectangle numbers are always even.

Solution

We need to prove that, for positive integer $n, n(n+1) is$ always even.

If n is even then (n+1) is odd, but (even)× (odd) is always even. If n is odd then (n+1) is even but odd × even is always even. So, rectangle numbers are always even.

Example 3: Prove that if you add two consecutive rectangle numbers, $U_n \ = n(n+1)$ and half the answer, the result is always a square number.

$U_n \ = n(n+1) = n^2 + n$

$U_{n+1} \ = (n+1)(n+2) = n^2 + 3n + 2$

So $U_n \ + U_{n+1}$=$2n^2 + 4n + 2 = 2(n^2 + 2n + 1)$

Half of this is =$n^2 + 2n + 1$

But this can be written as $(n+1)^2$ which is a square number- the result required.

Example 4: For any two integers a and b, if a < b, then $a^2 < b^2$. At first sight, the statement may appear to be true, but it is not.

A **counter example** is given by a pair, $a = -1, b = 0$. Indeed,

-1 <0 as required by the conditions of the statement. But the conclusion $1 = (-1)^2 < 0^2 = 0$ is obviously wrong. The amended statement that requires a and b to be positive, can be shown to be correct.

Example 5: Prove that if m and n are real and unequal, then

$$m^2 + n^2 > 2mn.$$

Solution:

$(m-2)^2 > 0 \ (by\ theorem\ (a \pm b)^2 > 0)$

But$(m-n)^2 = m^2 - 2mn + n^2 \ (Expanding)$

$\Rightarrow (m-n)^2 = m^2 + n^2 - 2mn > 0.$ (Associative Law of addition)

$$\Rightarrow \ m^2 + n^2 > 2mn. \quad (Q.E.D, or\ proved)$$

A flawed proof

Study the proof provided and explain what is wrong with the procedure.

Let $a = b$

$$Then\ a^2 = ab$$

$$a^2 + a^2 = a^2 + ab$$

$$2a^2 = a^2 + ab$$

$$2a^2 - 2ab = a^2 + ab - 2ab \ (\ subtracting\ 2ab\ from\ both\ sides).$$

$$2a^2 - 2ab = a^2 - ab$$

This can be written as $2(a^2 - ab) = 1(a^2 - ab)$

$$Dividing\ both\ sides\ by\ a^2 - ab\ \ gives\ 2 = 1.$$

Every step of a good mathematical proof is completely evident. Remember that a proof is simply a strong argument where each step is justified. Usually, proving involves several steps. It might not be immediately understood. John Mason, however, thinks that being stuck is a noble state and an important step in developing one's capacity for thought. Sometimes it's easy to become lost at the beginning, unsure of where to begin. It makes sense at this point to think of proving as a doable craft similar to other types of problem resolution. A few recommendations for ensuring the reliability of your evidence.

Steps:

1. Understand that mathematics uses information that you already know especially axioms or the results of other theorems.

2. Write out what is given, as well as what is needed to be proven Start with what is given, use other axioms, theorems, or mathematics that you already know to be true, and arrive at what you want to prove True understanding means you can repeat, and paraphrase the problem in at least 3 different ways pure symbols, flowchart, and using words.

3. Ask yourself questions as you move along "Why is this so?" and "Is there any way this can be false? Back up every statement with a reason! Justify your process.
4. Make sure your proof is step-by-step. It needs to flow from one statement to the other, with support for each statement, so that there is no reason to doubt the validity of your proof. It should be constructionist, like building a house: orderly, systematic, and with properly paced progress.

5. Ask your teacher or classmate if you have questions, it’s okay to ask questions every now and then- doing so is part of the learning process.

6. Designate the end of your proof. One way for doing this is writing Q.E.D. (quod erat demonstrandum, which is Latin for which was to be shown”). Technically, this is only appropriate when the last statement of the proof is itself the proposition to be proven.

7. Remember the definitions you were given. Go through your notes and the book to see if the definition is correct.

8. Take time to ponder about the proof. The goal wasn't the proof, it was the learning. If you only do the proof and then move on then, you are missing out on half of the learning experience. Think about it. Will you be satisfied with this?

Key ideas

Key I
• Deductive reasoning you start from a general statement you know is true and draw conclusions about a specific case • Mathematical proof is an argument that deduces the statement that is meant to be proven from other statements that you know for sure are true • In direct proof, the conclusion is established by logically combining the axioms, definitions, and earlier theorems

Reflection

• What are some of the experiences (i.e., cognitive, psychomotor, and affective) I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to help learners to apply their concepts deductive reasoning in the teaching and learning of mathematics? • How have my experiences in this training session prepared me to be a better classroom practitioner? Which specific examples can I draw from the course to support my position as a mathematics teacher?
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