Module for Bachelor of Education Programme (Primary)

EBS 277:
PSYCHOLOGICAL BASIS OF TEACHING AND LEARNING MATHEMATICS

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TABLE OF CONTENT

UNIT 1: PSYCHOLOGY OF MATHEMATICS

This unit introduces participants to the rationale for teaching and learning of mathematics and why prospective mathematics teachers should study this course. Finally, the unit ends with effective ways of teaching mathematics.

Learning outcome(s)

By the end of the unit, the participant will be able to:

1. explain the rationale for teaching and learning mathematics;
2. justify the need for studying psychological basis of teaching and learning mathematics;
3. discuss some effective ways of teaching mathematics.

SESSION 1: RATIONALE FOR TEACHING AND LEARNING MATHEMATICS

In this session, we will discuss the rationale for teaching and learning mathematics and some reasons for teaching and learning mathematics in school.

Learning outcomes

By the end of the session, the participant will be able to:

i. discuss the rationale for teaching and learning mathematics in school;
ii. reasons for teaching and learning mathematics in school.

Now read on …

Rationale for Teaching and Learning Mathematics

Mathematics is widely seen as very useful for everyday life, science, commerce and for industry. Mathematics enjoys a lot of recognition and respect from policy makers in many countries. In Ghana it is a core subject in basic schools, high schools and colleges of education. Mathematics is a logical, reliable and growing body of concepts, which makes use of specific language and skills to model, analyse and interpret the world. All students must have the opportunity and support necessary to learn significant mathematics with depth and understanding because mathematical knowledge enhances people's opportunities and options for shaping their futures.

Some reasons for Teaching and Learning Mathematics in School

1. Mathematics is a language understood across the world. It is known to be a means of communication which is powerful and unambiguous.

2. Mathematics is used by everyone in daily activities, be it at home, in office, workshop, market, etc. Everyone needs to develop mathematical concepts and skills to help them understand and play a responsible role in society. It is one of the essential areas of learning.

3. Mathematics helps to present information in diverse ways in the form of figures and variables in tables, charts, diagrams and geometrical or technical drawings. These can be systematically manipulated by combination and then applied to varied real life situations.

4. The strong mathematical competencies developed at the high school level are necessary requirements for effective study in mathematics, science, commerce, industry and a variety of other professions and vocations for students terminating their education at the high school level as well as for those continuing into tertiary education and beyond.

5. Mathematics is believed to train the mind. It helps in the development of the powers of logical reasoning and accuracy.

6. Mathematics is useful in the physical sciences, engineering, medicine, biological sciences, geography, commerce and industry. It forms the foundation for scientific development and modern technology. Mathematics education provides the opportunity for students to develop these skills and encourages them to become flexible problem solvers in all fields of human endeavour.

7. Mathematics is one of the core subjects in the school curriculum and so cannot be skipped. It is a compulsory subject to be taught and learnt at the basic and high schools. Teaching and learning of mathematics is an administrative/political fiat.

8. Mathematics can be studied just for fun. Mathematics is believed to hold a significant place in magic, puzzles and brain-cracking worlds. Teachers should endeavor to expose students to appropriate mathematics games, puzzles, brain teasers (Cockcroft's Report, 1982).

People may enjoy mathematics because of its usefulness. But it is far more likely that its appeal for us lies in the intellectual or aesthetic satisfaction that we derive from it. That is why one of the reasons for teaching and learning mathematics is that it is for fun. When you take the partial sums of the first few odd numbers 1, 3, 5, 7, 9, … you generate a sequence of numbers called square numbers which gives credence to the aesthetic appeal of mathematics.
Now observe what happens when we add the first two, first three, first four, … odd numbers:

$1 + 3 = 4$

$1 + 3 + 5 = 9$

$1 + 3 + 5 + 7 = 16$

$1 + 3 + 5 + 7 + 9 = 25$

……………………

Pause to reflect!
Look at the sequence of the partial sums: 4, 9, 16, 25, … These are the square numbers.
Let us conjecture and establish that:

The first two odd numbers add to give the second square number $2^2 = 4$.

The first three odd numbers add to give the third square number $3^2 = 9$.

The first four odd numbers add to give the fourth square number $4^2 = 16$.

The first five odd numbers add to give the fifth square number $5^2 = 25$.

Beauty of mathematics makes us 'feel' that the **first 100 odd numbers just have to add to $\mathbf{100^2}$**. No need for any rigorous proof but the power of pattern is such that you hardly feel the need to check that 10,000 is the correct answer. Teachers must capitalize on the use of patterns to make mathematics appealing to students and to make students see mathematics really as the study of patterns that have regularity.

Check this also! How many football matches are played altogether in a knockout tournament involving 512 teams? This is not an easy task especially for as many as 512 teams and more. But by the beauty of mathematics, we can consider just smaller numbers of teams and do pairing. For example, 4 teams in a knockout tournament to get 2 matches in first round and one match in the second round, giving $2+1=3$ matches altogether. Check for 8 teams, 16 teams and 32 teams to get 7, 15 and 31 matches respectively.

Now check for a pattern !!! 4 teams yield 3 matches; 8 teams yield 7 matches; 16 teams yield 15 matches, 32 teams yield 31 matches, … The trick is clear !!!

The number of matches played altogether is one less than the number of teams!!!

Predict now that there will be 63 matches altogether for 64 teams; 127 for 128 teams; 255 for 256 teams and 511 for 512 teams.

Further reflection!!! There is always only one team that should be unbeaten in all the rounds of the knockout tournament and that is the winning team; because any team beaten is out of the race. Assume that this team has played all the remaining teams and has beaten them. This explains why the number of matches played is one less than the number of teams involved.

In general we can say that for n teams involved in a knockout tournament, there will be $n-1$ matches played altogether.

This kind of exposure helps students to develop the habit of the mind to do a lot of mathematical thinking. This equips students to:

(i) show a lot of love for the subject and so often seek and create mathematics problems and try to solve them;

(ii) solve problems much faster than others;

(iii) memorize mathematical facts better than others.

Teachers should support students to develop mathematical thinking by:

a) arranging for them to experience mathematical processes rather than memorizing them;

b) ensuring that mathematical thinking occurs in all students;
c) making the development of mathematical processes a primary goal;
d) creating situations for students to solve problems and make discoveries in unfamiliar situations.

Key ideas

- All students must have the opportunity and support necessary to learn significant mathematics with depth and understanding because mathematical knowledge enhances people's opportunities and options for shaping their futures.
- Mathematics is one of the core subjects in the school curriculum, a compulsory subject to be taught and learnt at the basic and high schools in Ghana.
- The strong mathematical competencies developed at the high school level are necessary requirements for effective study in mathematics, science, commerce, industry and a variety of other professions and vocations for students terminating their education at the high school level as well as for those continuing into tertiary education and beyond.

Reflections

- What are some of the mathematical experiences that I went through at the basic/secondary/tertiary level(s)? How have these experiences prepared me to achieve the rationale for teaching mathematics at the basic school level?

Discussions

- Discuss **five** reasons why mathematics is taught in schools in Ghana.
- Discuss **four** strategies that teachers can use to support JHS students to develop mathematical thinking.

SESSION 2: WHY STUDY PSYCHOLOGICAL BASIS OF TEACHING AND LEARNING MATHEMATICS

This session will focus on why students study psychological basis of teaching and learning mathematics.

Learning outcome

By the end of the session, the participant will be able to explain the reasons for studying psychological basis of teaching and learning mathematics.

Now read on …

Importance of Studying this Course

The way school mathematics is often presented to learners partly accounts for the low performance of pupils in mathematics at the basic and higher school levels. Mathematics should be taught in such a way that it brings out the understanding of the process instead of the product. Prospective mathematics teachers should study this course (Psychological Basis of Teaching and Learning Mathematics) for the following reasons:

1. Mathematics teachers need knowledge of psychology to back the assertion that mathematics should be taught as a process and not as a product so that learners will understand and follow strategies to make them enjoy mathematics. The psychological theories give backing to the teaching strategies often employed in the classroom for effective teaching of the subject.

2. Learning in general deals with the use of the intellect and mathematics is an intellectual activity. Psychological principles to be learnt in this course equip the teacher to understand how students learn mathematics or how the human brain functions. Teachers need to know how experience and human thinking interact to bring about mathematical ability.

3. The *individual differences* among the learners we teach are highlighted in the theories of learning covered in this course. Mathematics teachers rely on sound theories to teach the subject well to the heterogeneous group of learners in the classroom. Learners have different learning abilities and difficulties and also different learning styles and teachers need to cater for all as best as possible. The need to design strategies to cater for all these categories of learners call for the study of the psychological basis for mathematics instructions.

4. There is the need for a psychological framework based on empirical evidence from the behaviour of learners in the classroom. The course provides some basis for our views on teaching. The knowledge of the psychological principles enables mathematics teachers to help those around us, even fellow teachers in the field to accept new innovations in the teaching and learning of mathematics.

Key ideas

- Mathematics teachers rely on sound psychological theories to teach the subject well to the heterogeneous (i.e., learners with different abilities and difficulties) group of learners in the classroom.
- Mathematics teachers' knowledge of the psychological principles enable them to help students and sometimes colleague teachers to accept and apply innovations in the teaching and learning of mathematics.
- Prospective mathematics teachers should study psychological basis of teaching and learning mathematics at the training stage.

Reflections

- How has my knowledge of psychological theories of learning mathematics helped me to effectively present mathematics lessons in my JHS classroom?

Discussions

- Discuss **four** reasons why prospective mathematics teachers should learn about psychological basis of teaching and learning mathematics.

SESSION 3: EFFECTIVE TEACHING OF MATHEMATICS

In this session, we will focus on the effective ways of teaching mathematics. Also, conditions under which effective teaching of mathematics thrives will be highlighted.

Learning outcome

By the end of the session, the participant will be able to explain effective ways of teaching mathematics and the conditions under which effective teaching and learning of mathematics thrives.

Now read on …

What is Effective Teaching of Mathematics?

Effective mathematics teaching is the teaching that requires the teacher to understand what *learners know* and *need to learn* and then *challenge and support* them to learn it well (NCTM, 2000). The amount of knowledge learners gain depends largely on the experiences that the teacher provides everyday in the classroom. Teacher's actions encourage learners to think, question, solve problems and discuss ideas, strategies and solutions. Teachers are to ensure that they pose worthwhile and challenging mathematical tasks to enable students to think critically and logically. Students on their part are to learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. Mathematics requires not only computational skills but also the ability to think and reason mathematically in order to solve the new problems and learn the new ideas that learners will face in future. Learning should be an *active endeavour of learners* that takes place best in an environment that stresses problem solving, reasoning and thoughtful interaction among learners. The learner must be seen "*doing*" mathematics in the classroom which involves both the teacher and the learners engaged in *exploring, investigating, conjecturing, solving, justifying, representing, formulating, discovering, constructing, verifying, explaining, predicting, developing, describing, etc.* These are signs that learners are active participants and this makes them capable of making sense of the world and to become confident "*doers*" of mathematics. Mathematics then makes sense to them and they believe in their ability to make sense of mathematics. Teachers should therefore select mathematical tasks that are relevant to the learners' world.

Mathematics teachers should make students come to recognize the fact that the most *basic idea* in mathematics is that mathematics makes sense by ensuring that:

(i) everyday students experience that mathematics makes sense;
(ii) students come to believe that they are capable of making sense of mathematics;
(iii) teachers stop teaching by telling and start letting learners make sense of the mathematics they are learning;
(iv) teachers believe in all their students.

Effective mathematics teaching thrives well in a classroom environment where:

(i) doing mathematics is *not threatening*;
(ii) every learner is *respected* for his or her ideas;
(iii) learners can feel *comfortable taking risks,* knowing that they will not be ridiculed.

Learners must be encouraged to actively figure things out, test ideas and make conjectures, develop reasons and offer expectations. Learners should work in groups, in pairs, or individually, but they are always sharing and discussing, defending their methods and justifying their solutions. A teacher can succeed in creating this environment if he or she has a personal feel for doing mathematics in this manner. Worthwhile tasks posed in a conducive mathematics environment makes students feel that mathematics makes sense, believe they can do mathematics and make sense of it, and that you believe in their ability to do mathematics. Teachers should be more of facilitators to the students, listening to them, giving hints where necessary and allowing them to explore and investigate and make their own discoveries.

Effective mathematics teachers are cognizant of the following facts about *mathematical thinking* and how to help their students develop mathematical thinking.

1) Mathematical thinking is an act of sense-making and rests on the processes of specializing and generalizing, conjecturing and justifying.

Teachers should involve students in the process of mathematical enquiry by:

a) presenting a problem for the students to work on.
b) eliciting students' conjectures about the general form of the concept based on specific cases.
c) withholding judgement to maintain an authentic state of uncertainty regarding the validity of conjectures.
d) asking students to test conjectures and justify them to their peers. Do not reject students' initial conjectures, offer a counter-example.

   Be interactive and try to acknowledge and validate authorship of students by labelling their contributions/conjectures by their names; e.g., 'Alima's conjecture'.

2) Habits of individual reflection and self-monitoring often accompany processes of enquiry in mathematics. Teacher's questions are very useful in prompting self-directed thinking among students. Pose worthwhile mathematics tasks.
3) Mathematical thinking develops through teacher scaffolding of the processes of enquiry. Help students to make sense of the mathematics they are learning by prompting them to classify, elaborate, justify and critique.
4) Mathematical thinking can be generated and tested by students through participation in equal status partnerships. Encourage social interaction among peers. Establish classroom norms with the students and encourage small group work and class discussions.
5) Interweaving of familiar and formal knowledge helps students to adopt the conventions of mathematical communication. Avoid the use of technical terms until students have developed an understanding of the underlying mathematical ideas. Make the precise language available to them at the appropriate time. This makes their mathematical thinking visible while engaged in discussion with their mates.

Because students learn mathematics by making sense of it themselves, teachers should:

i. engage students actively in the lessons, focusing on long-term perspectives;
ii. model mathematical thinking and encourage students to make and evaluate conjectures;
iii. encourage communication between students so they can learn from each other, sharpen their understanding, and practise using the specific language of mathematics.

Key ideas

- Effective mathematics teaching is the teaching that requires the teacher to understand what learners know and need to learn and then challenge and support them to learn it well (NCTM, 2000).
- Effective mathematics teaching thrives well in a classroom environment devoid of threats, respect individual learner's ideas and where learners can feel comfortable to take risk.
- Teacher's actions encourage learners to think, question, solve problems and discuss ideas, strategies and solutions.

Reflections

- How has my exposure to the content of this session prepared me to effectively teach mathematics in the classroom?

Discussions

- Discuss any **four** facts about mathematical thinking that effective mathematics teachers must be cognizant of and employ in their teaching of mathematics.
- Explain the main components of effective teaching of mathematics.
- Explain **five** facts about mathematical thinking and the role of the teacher in helping students to develop mathematics thinking.

SESSION 4: CHILDREN'S ERRORS AND MISCONCEPTIONS IN MATHEMATICS

In this session, we will focus on children's errors and misconceptions in learning mathematics.

Learning outcome

By the end of the session, the participant will be able to discuss some errors and misconceptions that children have when learning mathematics.

Now read on …

What is a misconception?

Children usually draw on their own understanding of the world around them and unavoidably misconceive some ideas. A young child referring to a cow and say 'horse' has misconceived the being as a horse: (a) an animal coloured white and black in a field; (b) eating grass; and (c) swishing its tail. All of these attributes satisfy definition of a horse. Later the child will be able to distinguish between a horse and cow. This development has to be mediated by the teacher.

The term '*misconception'* is commonly used when a learner's conception is considered to be in conflict with the accepted meanings and understandings in mathematics (Barmby et al .., 2009). It could be the misapplication of a rule, an over-generalization or under-generalization, or an alternative conception of the situation. For example, a *number with two digits is smaller than a number with three digits* works in some situations (e.g. 25 is smaller than 237) but not necessarily in others where decimals are involved (e,g, 25 is not smaller than 2.37).

Misconceptions can become rigid and resistant to revision later on and so teachers must be aware of potential misconceptions, the possible reasons why they have developed and provide the appropriate mediation.

An *error* in mathematics could occur as a result of carelessness, misinterpretation of symbols or text, lack of relevant experience or knowledge related to that mathematical learning objective or concept, a lack of awareness or inability to check the answer given, or the result of a misconception. Mathematical errors are principally formed within surface levels of knowledge. A child's response to a task is procedural and can be corrected by the teacher providing correct alternatives. The nature of mathematical tasks selected by the teacher has potential for children making errors and so teachers need to consider the complexity of the task so as to make it *sufficiently challenging* and not *too challenging*.

Consideration must also be given to the way the task is presented and the ability of the child to translate the task so as to ensure whether the pupil knows what is required in mathematical terms. Sometimes errors can be worsened by teachers when they assumed too much about children's experiences. Incorrect uses of resources can also lead to children making errors. Some mathematical errors could be avoided by teachers' awareness, skillful choice of task and clarity of explanation.

Dealing with a child's mathematical error demands skills of **diagnosing** and the handling depends upon the nature (and frequency) of the error observed. Misconceptions are far more problematics than errors. **Diagnosis and dialogue** are often needed to ascertain the misconception, which can be time- consuming. Misconceptions should be regarded as *evolving understandings in mathematics … essential and productive for the development of more sophisticated conceptions and understanding.*

Mathematical errors can provide a useful insight for teachers into a child's thinking and understanding, an effective mechanism for assessment for learning and can enable children to learn from mathematical mistakes. It appears far more productive for teachers to investigate the reason a child provides a given answer. Teachers should adopt a *constructive attitude to their children's mistakes* and children should recognize that analysis and discussion of mistakes or misconceptions can be helpful to their mathematical development. Placing children in situations where they feel in control of identifying mathematical errors/misconceptions leads to greater openness on the part of the children to explore and discuss their own misconceptions. This approach has an underlying belief that children's mathematical understanding is more likely to be developed if children are given opportunities to explain their thinking and to compare their thinking with that of peers and teachers.

Key ideas

- The term '*misconception'* is commonly used when a learner's conception is considered to be in conflict with the accepted meanings and understandings in mathematics (Barmby et al ., 2009).
- An *error* in mathematics could occur as a result of carelessness, misinterpretation of symbols or text, lack of relevant experience or knowledge related to that mathematical learning objective or concept, a lack of awareness or inability to check the answer given, or the result of a misconception.
- Mathematical errors can provide a useful insight for teachers into a child's thinking and understanding, an effective mechanism for assessment for learning and can enable children to learn from mathematical mistakes.
- Teachers should adopt a *constructive attitude to their children's mistakes* and children should recognize that analysis and discussion of mistakes or misconceptions can be helpful to their mathematical development

Reflections

- How has my reading of the content of this session equipped me with the knowledge and strategies to recognize and deal with children's misconceptions and errors?

Discussions

- Explain **three** reasons why children fail mathematics
- Distinguish between misconception and errors in mathematics.
- Discuss **two** strategies the teacher can employ to help pupils to overcome errors and misconceptions in mathematics.

UNIT 2: THEORIES OF LEARNING

This unit introduces participants to some theories of learning. It focuses on the behaviourist theory of learning, cognitivist theory of learning and constructivism.

Learning outcome(s)

By the end of the unit, the participant will be able to explain:

i. behaviourist theory of learning;
ii. cognitivist theory of leaning;
iii. constructivist theory of learning.

SESSION 1: THEORIES OF LEARNING

In this session, we will focus on the meaning of theories of learning.

Learning outcome

By the end of the session, the participant will be able to explain learning and learning theories.

Now read on …

Learning is the process of creating knowledge, or the process of absorption of knowledge resulting from the interaction between the teacher and the learner. Learning is said to have occurred when there is a relatively permanent change in behaviour acquired through an experience. Learning is usually directed towards specific goals through organized patterns of experience. Kolb (1984) identified three essential characteristics of learning.

(i) Learning is a continuous process based on experience.
(ii) Learning is a holistic process of adaptation to the world.
(iii) Learning involves transaction between the person and the environment.

Meaning of Theories of Learning

A theory is a reasonably or scientifically accepted explanation for a fact or event. Theories are general principles and knowledge. Theories are more powerful, abstract and more general than common-sense models. They are sometimes used to predict events which are contrary to common sense, like iron ship floating on sea. They are mental models which increase our power to understand the invisible causes behind visible events. Thus as teachers, we need to intervene in the mental processes of our growing children.

Learning theories are therefore conceptual frameworks that describe how information is absorbed, processed, and retained during learning. Learning theories help in planning instruction and developing curriculum materials to illustrate. They provide a strong argument for using appropriate models and concrete materials to illustrate mathematical concepts and establish bridges. Learning mathematics in particular, becomes a change in behaviour brought about through brain action or thinking. This occurs more appropriately when students are made to face situations that call for making discoveries, abstractions, generalizations and organizations in mathematics. Mathematics becomes functional in the lives of children when they have developed basic computational skills and can apply mathematics to their world.

Some theories hold the view that learners must be actively involved in the learning process, both mentally and physically, so that they can truly benefit from a given experience. Other theories of learning portray the learner as a passive recipient in the learning process, almost as if the learner's mind were a blank slate that could be written on at will by some outside source. Some theorists view the role of the teacher primarily as a guide or facilitator of learning, one who effectively organizes the conditions under which the learning can take place and then exposes children to those conditions. Other theorists have depicted the role of the teacher as the prime expositor of knowledge, and as the person primarily responsible for children's learning, almost as if the children were not intimately involved placing the responsibility for learning more on the individual learner rather than on the teacher.

Key ideas

- A theory is a reasonably or scientifically accepted explanation for a fact or event.
- Learning theories are conceptual frameworks that describe how information is absorbed, processed, and retained during learning.
- The beliefs held by a particular learning theory determine the role of the teacher and the learner in the teaching and learning process.
- Some theories hold the view that learners must be mentally and physically active in the learning process.
- Other theories of learning portray the learner as a passive recipient in the learning process, almost as if the learner's mind were a blank slate that could be written on at will by some outside source.

Reflections

- How has my knowledge of the theories of learning influenced my teaching and general classroom discourse?

Discussions

- Explain any **three** features of theories.

- Explain why different learning theories differ on the roles of teachers and learners in the classroom.

SESSION 2: BEHAVIOURIST THEORY OF LEARNING

This session concentrates on the behaviourist theory of learning.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain the behaviourist theory of learning;
ii. explain the learning hierarchies of Robert Gagné.

Now read on …

Behaviourist Theory of Learning

To the Behaviourists, learning is a connection between stimulus and response or between response and reinforcement. Behaviourism has its root in stimulus-response (S-R) learning and conditioned learning. They see a change in behaviour as the result of an individual's response to an event (stimulus) that occurs in the environment. Early behaviourists (like Edward L. Thorndike, B. F. Skinner, Pavlov and Robert Gagne´) conducted experiments to support this theory. The Russian psychologist, Ivan Pavlov was known to have conditioned a dog to salivate to a sound of a bell (in classical conditioning). This form of learning is believed to be the same whether in dogs or in humans. The American psychologist, Skinner, carried out experiments where rats and pigeons were enticed to press or peck a lever in order to obtain pellets of feed as examples of operant conditioning. These were meant to gradually shape an appropriate behaviour into a desired outcome. This was criticised as workable for animals and not really for human beings.

Behaviuorists generally see learning as an aspect of conditioning and so they advocate a system of rewards and targets in education, concluding that behaviour can be shaped through reinforcement (reward and punishment). **Reinforcement** refers to any event, which increases the frequency of the response it follows. Reinforcement such as punishment is also intended to redirect pupils' behaviour from an undesirable one to a desired behaviour. Rewards reinforce responses. Frequent rewards make responses stronger. Praise and other rewards follow a favourable result, i.e, when a desired change in behaviour occurs in a child. Punishment and scolding follow an unfavourable result. A reward increases the likelihood of the behaviour recurring, a punishment decreases its likelihood. Thus, pleasurable consequences strengthen behaviour while unpleasant consequences weaken it.

Behaviourists hold the view that the child's mind is empty (is a tabula rasa) and needs to be filled by teachers. Teaching is something done to the child – a process through which a child is put to obtain a desired behaviour. The implication here is that education must produce changes in children by making them act in response to situations organized by the teacher and to act in similar manner to situations outside school.

This theory is described as simplistic, provides instructional guidance, allows for short-term progress and lends itself well to accountability pressures. Its disadvantage is that it places emphasis on the short term objectives and this is likely to dominate instructional planning. For this reason, it should not be exclusively relied on.

Robert Gagné and Learning Hierarchies

Robert Gagné is an advocate of Behaviousisim who attempted to clarify the relationships between the psychology of learning and instruction. This has to do with arranging the conditions to bring about the most effective learning of the five **"categories of capabilities**" identified as *intellectual skills, cognitive strategies, verbal information, motor skills, and attitudes*.

The central question often posed is "What do you want the individual to be able to do?" Gagné and his colleagues rely on *task analysis* by breaking concepts into smaller bits and pieces on the assumption that it is possible to subdivide a desired learning goal into its constituent parts and that once these parts have been learned, they will be synthesized by the learner in such a way that the larger goal is understood. Their belief is that the whole (the desired goal or terminal capability) is equal to the sum of its parts (the identified component parts).

Teaching and learning should be very specific or goal directed and be based on task analyses. Once the learning objective has been established, it must then be broken into component parts. Students' understanding is then assessed to determine which students possess which prerequisite behaviours. When a pupil lacks one or more of these, he/she must be taught those skills before the desired learning goal (terminal capability) can be reached. This culminated in what came to be known as *Learning Hierarchies.*

Suppose the desired capability is "adding two unlike fractions". First ask the question, "What does the student need to know in order to add two unlike fractions?" If you identify the immediate pre-requisite to be "Adding two like fractions", then the next question is "What does the student need to know in order to add two like fractions?" You keep asking similar questions until the basic concept is attained. This gives rise to a "*Map of pre-requisites*" with the desired capability as the apex of the pyramid.

Gagné was mainly concerned with the outcome of the educative process specified in terms of what the learner can do. His focus was on the "**what**" of the learning process but not with "**how**" the child actually learns or with what behaviour of the teacher. Gagné's model suggests that learning be organized as a series of guided exercises, giving rise to programmed learning and drill and practice. He suggested that information should be presented in small amounts so that responses can be reinforced. Behaviourists insist on stating educational objectives in specific behavioural terms and this has formed the basis for much of the school mathematics curricula. The attainment of the desired capabilities becomes the major goal of the educational process. There is the temptation to involve students in those (and perhaps only those) capabilities that are directly related to the main capability and which are likely to lead to student achievement. Critics state that because the objectives (capabilities or goals) set thus determine the child's activities, this is likely to result in very limited opportunities for deviation from the development of the desired capabilities.

The *danger* inherent in this approach is the possible elimination of informal kinds of learning activities, which may not directly contribute to the attainment of a specific capability, but ultimately may prove vital in the overall learning process. There is much evidence that this has happened in the majority of our schools where calculations with paper and pencil have dominated the mathematics curriculum, and exploratory and more loosely structured activities are almost nonexistent. Secondly, the notion that, some very desirable higher order capabilities, like problem solving, simply do not lend themselves to task analytic procedures. Gagné himself has suggested that task analysis is more appropriate to lower level objectives. Thirdly, there is also the probability that important capabilities are often overlooked.

The approach tends to inhibit transfer of training because students learn specific concepts/skills well and these specifics in turn act as a source of interference (negative transfer) for new and different situations. The students are most likely to develop only a limited capability for transfer of training. Because specific concepts/skills are taught directly, transfer of training is also assumed to be quite specific.

The fact that Gagné lays much emphasis on the *what* of the learning process and not with the *how* does not follow that teachers using this approach use the lecture or expository approach as the sole teaching technique. The child may be taught by lecture, or discussion, or even discovery. The child is not necessarily passive (listening). The pupil's actions are somewhat confined under the very strict goal-directed procedures.

Teaching for specific knowledge, using programmed learning, is most effective in short-term specific learning situations, such as the development of computational skills and less appropriate for higher order objectives. This will however, continue to have significant impact on curriculum development in the area of school mathematics because there is logic in Gagné's arguments. Learning should not be left to unattended or unanticipated occurrences.

The following are some of the roles of the teacher in **applying** Behaviourism in the classroom.

(i) Teacher reinforces the correct responses and discourages wrong responses.

(ii) Feedback to learners is provided immediately. Feedback to students is seen as a reward.
(iii)Teacher heavily controls the stimuli, chooses the correct response and provides the appropriate reward. Teacher decides which response is most appropriate to a particular task.
(iv)Teacher mostly uses marks, prizes and praise as different reinforcements.
(v) Lessons are usually arranged into series of graded steps. That is, programmed learning activities are employed.
(vi)Teacher provides opportunities for practice. Drill and Practice dominates.

Key ideas

- Behaviourism has its root in stimulus-response (S-R) learning and conditioned learning. They see a change in behaviour as the result of an individual's response to an event (stimulus) that occurs in the environment.
- Behaviuorists generally see learning as an aspect of conditioning and so they advocate a system of rewards and targets in education, concluding that behaviour can be shaped through reinforcement (reward and punishment).
- Behaviourists insist on stating educational objectives in specific behavioural terms and this has formed the basis for much of the school mathematics curricula.
- The concept of behavioural objectives is based on Gagné's task analysis.
- Gagné's model lends itself to the programming of learning sequences, where the parts of the sequence are the sub-behaviours that underlie the larger objective.

Reflections

- How has the learning of this session influenced me to employ good behavioural practices to support the teaching and learning of mathematics in the classroom?

Discussions

- Explain the term reinforcement and the various forms of reinforcement.
- Explain the major contributions of Robert Gagne' to mathematics instruction in a typical mathematics classroom.
- Identify **three** roles of the mathematics teacher from a behaviourist point of view.

SESSION 3: COGNITIVE (DEVELOPMENTALISTS) THEORY OF LEARNING

This session discusses the cognitivist (developmentalist) theory of learning. Attention will be given to the works of Jean Piaget, Jerome Brunner and Zoltan Dienes.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain the Cognitive theory of learning;
ii. explain Dienes' principle of variability;
iii. explain Bruner's phases of instruction;
iv. explain Piaget's cognitive stages and conservation ideas.

Now read on …

Cognitive (Developmentalists) Theory of Learning
The Cognitive theory is concerned with both *how* children learn and with *what* it is that they learn. It places great emphasis on the process dimension of the learning process. The theory provides the major theoretical rationale for the promotion of active student involvement in the learning process. The advocates give highest priority in the teaching and learning process. They hold the view that, learning is a very personal matter and that true understanding involves an internalization of concepts and relationships by the individual involved. The main focus is on the interrelationships between parts as well as the relationship between parts and the whole. To them the whole is greater than the sum of its parts and that the learning of large conceptual structures is more important than the mastery of large collections of isolated bits of information. Learning is intensely personal in nature and so it is intrinsic to the individual. The meaning that each individual attaches to an experience is of great importance. When individuals are encouraged to interact personally with various aspects of their environment, the degree of meaning obtained is maximized. It is the physical action on the part of the learner that contributes to his or her understanding of the ideas encountered. Manipulative materials when used properly can promote these broad goals.

The Cognitive theory builds on two key **underlying assumptions**: (i) the memory system as an active organized processor of information and (ii) prior knowledge plays an important role in learning. The following **principles** emerge from the assumptions that underlie Cognitive learning theory.

1) Learning is a personal experience and teachers should endeavour to serve as facilitators.
2) Learning is something that happens as a result of thinking. Teachers must consider the mental processes of the learner and expose them to appropriate activities that promote thinking.
3) Learners have to attain certain stages of cognitive development before they are introduced to certain mathematical concepts. This is because children cannot learn the same content as adults and even among children there are individual differences to consider.
4) Learning takes place when learner mentally reorganizes his/her inner world of concepts and interprets the environment as such. Learning consists of changes in mental constructs and processes.
5) Learning is a process in which the student is actively engaged. Learner is an active agent in the learning process attempting to process information fed into him. Learner is not a mechanical product of the environment.
6) Meaningful learning takes place when new information fits into existing cognitive structures. This calls for teachers to identify appropriate prerequisites.

In a classroom where cognitivism is **applied**:

1) the teacher shows a good knowledge of student's prerequisite knowledge;
2) students are assisted in developing meaning by providing puzzles and rules to work through;
3) the teacher provides a structure or helps the student to create a structure to which is added new learning;
4) inductive-deductive approach of teaching dominates. Inductive teaching method is the process that starts with specific situations leading to the general and is usually based on specific experiments or experiential learning exercises. Deductive teaching method on the other hand progresses from the general concept to the specific use or application.

Jean Piaget's Stages of Cognitive Development and Conservation Ideas

Piaget was a Swiss psychologist who was of the view that teaching and learning involve taking the environment apart, physically or mentally and reconstructing it. Piagetian theory indicates that in learning, children pass through developmental stages and that the use of active methods which gives scope to spontaneous research by the child helps him or her to rediscover or reconstruct what is to be learned "not simply imparted to him". Every normal student is capable of good mathematical reasoning provided attention is directed to activities of the student's interest and the emotional inhibitions that too often give him/her a feeling of inferiority in mathematics lessons are removed.

Piaget held the belief that learning takes place in relation to the relevant stage of development, but it is achieved through interaction with the environment. His theory of intellectual development views intelligence as an evolving phenomenon occurring in identifiable stages that have a constant order. He identified three learning processes: (i) formation of mental concepts, (ii) adaptation of these concepts in the light of experiences, and (iii) relating the concepts to form structures. He regarded intelligence as effective adaptation to one's environment which involves the continuous organization and reorganization of one's perceptions of, and reactions to, the environment. Adaptation is the essential ingredient for learning, which takes place through the complementary processes of *assimilation* (fitting new situations into existing psychological frameworks) and *accommodation* (modification of behaviour by developing or evolving new cognitive structures). The cycle of the assimilation-accommodation should be used continually to restore equilibrium to an individual's cognitive framework. Equilibration is a balance between assimilation and accommodation and this is attained as the individual organizes the demands of the environment in terms of previously existing cognitive structures. Movement of a child from one stage of cognitive development to another occurs through the process of equilibration, through understanding the underlying concept so that the understanding can be applied to new situations.

Piaget believes that human intellectual development progresses chronologically through four sequential stages: sensory-motor, pre-operational, concrete operational, and formal operational. The order in which the stages occur has been found to be largely invariant, however the ages at which people enter each higher order stage vary according to each person's hereditary and environmental characteristics, the degree of meaningful social and educational transmission, and the nature and degree of relevant intellectual and psychological experiences. The children we teach should therefore not be seen and treated as little adults.

Sensori-motor stage (0-2 years of age) is where the child begins to use imitation, memory and thought. The child moves from reflex actions to goal-directed activity. He learns about the concept of *permanence* of objects and that things exist only when he sees or touches them. Child plays endless games of handling a toy to someone as long as the toy will be handed back.

Preoperational stage (2-7 years) is where children gradually develop language and the ability to think in symbolic form. The period is characterized by child's growing power of representation. Child grows to be confident that the world is as he perceives it. Child's perception of the world is egocentric. Child has difficulty seeing another person's point of view. Child can discover by experiment that $7 - 4 = 3$, but may not be able to relate it to $3 + 4 = 7$. This gradually changes by the end of the period.

Concrete operational stage (7-11/12 years) is where children are able to solve concrete (hands-on) problems in logical fashion. There is growing application of logic to physical situations, real or imaginary. The concept of reversibility deepens. They understand the laws of conservation and are able to classify and seriate.

Formal operational stage (11/12 years or more) is where children are able to solve abstract problems in logical fashion. They can argue from abstract hypotheses and to make deductions solely on the basis of logic. Their thinking becomes more scientific, they develop concerns about social issues and about identity.

Teachers are expected to use active teaching methods that permit the child to explore spontaneously and require that "new truths" be learned, rediscovered or at least reconstructed by the student. The teacher is mainly a facilitator and organizer who creates situations and activities that present a problem to the student. When a student achieves a particular knowledge through free investigation and spontaneous effort the student will better be able to retain it at a later date. Emphasis is placed on the crucial role that student-to-student interaction plays in both the rate and the quality with which intelligence develops.

Conservation Ideas

Piagetian conservation tasks have proved useful in providing one measure of the intellectual level at which children are operative. Children normally conserve number, length, and volume between the ages of 6 and 8, though with some exceptions. Children should have the opportunity to handle the materials and should always be asked to justify or explain their conclusions.

Conservation of Number

Conservation of number refers to the ability of a child to realize that the number of members in a collection remains unchanged even as the members of the collection are physically rearranged. Two empty containers, A and B, are placed before the child and filled with the counters one at a time. Teacher should ensure that each time a counter is placed in container A, a counter is *simultaneously* placed in container B. After each of 5 or 6 pairs of counters are so placed, teacher asks the child whether the two containers hold an equal number of counters for the child to establish equality but not necessary to count them. Now pour the contents of one of them onto the desk top, spreading them over a large area. Then ask, “Are there just as many counters on the table top as there are in the other container?” Or, “are there more counters, or less counters on the table top than in the other container?” Ask the child to explain. If the child is able to say that, there are as many counters on the table top as there were in the container, then the child is said to have ***conserved number***.

Conservation of length has to do with the child being able to recognize that the length of an object remains the same no matter the different positions or inclinations. The length of the pairs of sticks A, B, and C are the same no matter the orientation once the equality of lengths have been established in diagram A.

Conservation of Volume refers to the ability of a child to determine whether volume (in this case amount of liquid in a container) is unchanged under a certain type of physical rearrangement. The experiment involved filling two identical containers, A and B, to the same level. Pour the contents of one of the containers into a third container, C, having a smaller cross section and ask again “Is there just as much water in container C as there is in container A?” Ask the child to explain.

Jerome Bruner and Phases of Instruction

Bruner was interested in the general nature of cognition (conceptual development), and so has provided additional evidence suggesting the need for firsthand student interaction with the environment. His widely quoted view that “any subject can be taught effectively in some intellectually honest form to any child at any stage of development”. That is, any body of knowledge can be presented in a form simple enough so that any particular learner can understand it in recognizable form. Bruner’s theory suggests taking every learning task through three sequential phases: concrete to semi abstract to abstract. The proportion of time to spend on each phase will vary according to age, experience and ability of the learner.

Bruner’s instructional model is based on four key concepts: *structure, readiness, intuition, and motivation*. Bruner suggested that teaching students the structure of a discipline leads to greater intellectual involvement as they discover basic principles for themselves. He stated that learning the structure of knowledge facilitates comprehension, memory, and transfer of learning. The idea of structure in learning leads naturally to a process approach in which the very process of learning (*how* one learns) becomes as important as the content of learning (*what* one learns).

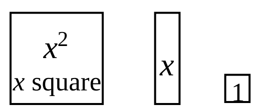
The model suggested that one can experience and subsequently think about a particular concept on three different levels: enactive, iconic and symbolic. Learning at the *enactive* level involves hands-on or direct experience that is, being able to perform coordinated movements without being able to express it. The student is engaged in first hand manipulating, constructing, or arranging of real-life objects. The student interacts directly with the physical, representing the concrete phase. The strength of enactive learning is its sense of immediacy.

Learning at the *iconic* level (imagery phase) is characterized by the use of visual medium dominated by visual and perceptual organizations like films, pictures, diagrams. This is the semi concrete level where learner can make mental pictures of concrete objects and can perform imagery “maps” in the mind.

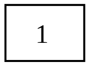
The *symbolic* level also referred to as the *abstract* phase is where learner manipulates and/or uses symbols irrespective of their enactive or iconic counterparts. The learner is capable of using abstract symbols to represent reality. Ideas are mostly represented in symbols and learner can make mental manipulations on abstract symbols like $(x+2)^2 = x^2 + 4x + 4$.

Bruner experimented with children using square blocks/tiles to teach binomial expansion even at a very early stage of cognitive development of the learner.

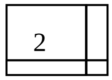
1. Learners were first introduced to three kinds of flat pieces of wood or “flats” –
   (i) unknown square or *x* square;
   (ii) 1*x* or just *x*;
   (iii) 1 by 1 called 1.



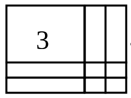
2. Learners were allowed to play freely with these materials (enactive free play stage)
3. Teacher now poses a problem*: Can you make larger squares than this ‘x square’ by using as many of these smaller pieces as you want?*
   With the materials available, this was easily carried out by learners.
   They quickly made the second square and then the third one as shown in the diagram.
4. **Teacher:** Can you describe what you have done?
5. Expected response: We have one “ *x square* with two (*x* by 1) and a 1.
   Teacher: Keep a record of what you have done.
   Teacher may introduce notation $x^{\Diamond} + 2x + 1$, [introducing plus for *and*]



$\rightarrow \quad x^{\square} \qquad \rightarrow x^2$



$\rightarrow (x + 1)^{\square} \rightarrow x^{\square} + 2x + 1 \rightarrow x^2 + 2x + 1$



$\rightarrow (x + 2)^{\square} \rightarrow x^{\square} + 4x + 4 \rightarrow x^2 + 4x + 4$

6. The second description could be by the dimensions of the new square formed – an x and a 1, (x + 1) on each side to get [(x + 1) by (x + 1)] or $\left[(x+1)^{\lozenge}\right]$.
   Since they both describe the same square learners can conclude that
   $$(x+1)(x+1) = (x+1)^{\lozenge} = x^{\lozenge} + 2x + 1$$
7. Learners are now allowed to make larger and larger squares and generate the notation for each of them.
8. The expectation is that learners will begin to see a pattern and regularities. E.g. the x's progress : 2, 4, 6, 8, - - - ; the 1's progress 1, 4, 9, 16, - - - **[Do – Talk – and Record]**

By the use of appropriate leading questions learners are led to do this discovery. They will sense that there is a pattern to be discovered and will search for it.

What is most important for teaching basic concepts is that the learner be helped to pass progressively from concrete thinking to the utilization of more conceptually abstract ideas. A key to **readiness** for learning is intellectual development, or an enlarging perspective of how a child views the world. This calls for a rich and meaningful learning environment, coupled with an exciting teacher who involves children in learning as a process that creates its own excitement. There is emphasis on the idea of intrinsic motivation or learning as its own reward. Readiness thus depends more on an effective mix of these three learning modes.

Bruner's work suggests that these modes should be interactive in nature, freely moving from one mode to another. For example, given the equation $x^2 + 2x + 1$, the child could be asked to draw a picture of this situation or to model this using the flats. This would in effect be a translation from the symbolic ($x^2 + 2x + 1$) to the iconic mode and to the enactive mode.

Every teaching targets learning. Teachers must therefore consider the following during mathematics instructions: a) the predisposition of the child towards learning mathematics, b) the way the knowledge to be learnt is structured, c) the sequence of presenting the knowledge, d) the nature of motivation and rewards provided. *Predisposition* deals with the fact that the will for a child to learn is ingrained. Children are curious to learn. This intrinsic curiosity must be channeled by teachers into guided discovery, where experiences are carefully selected and represented in language, pictures and symbols. *Structure of knowledge* suggests that mathematics tasks selected for children to do must *cry out* for simplification into conceptual form. Addition tasks like (3 +

$4), (3 + 3); (3 + 5), (5 + 3); \ldots$, can be seen to *cry out* for the conceptual conclusion that addition is commutative. *Sequence of presentation* of mathematical knowledge influences how children learn. Because of individual differences, teachers need to help children build different paths to the same goal. Bruner advocates for a mathematics curriculum which is spiral where mathematical concepts are first presented in an imprecise but honest language, and then later revised and described more precisely. *Motivation and rewards* are to take the intrinsic form. Children should be aware that their new learning is leading to a useful goal. They should be encouraged to correct their own work using say a calculator to confirm, a pattern tor any appropriate approach to confirm. This gives intrinsic satisfaction to children.

Implications of Bruner's Work

Mathematics textbooks are exclusively iconic and symbolic containing pictures of things, and symbols to be associated with those things. Mathematics programmes that are dominated by textbooks are inadvertently creating a mismatch between the nature of the learner's needs and the mode in which mathematical content is to be assimilated or learned. Cognitive psychologists have indicated that (1) knowing is a process, not a product; (2) concepts are formed by children through a reconstruction of reality, not through an imitation of it; and (3) children need to build or construct their own concepts from within rather than have those concepts imposed by some external force. Bruner’s enactive-iconic-symbolic model fits this.

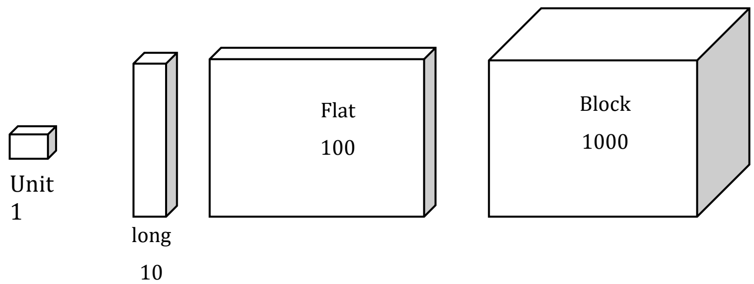
Learners need a large variety of enactive experiences. Their concepts basically evolve from direct interaction with the environment. A mathematics programme more compatible with the nature of the learner should include more manipulative materials and more experiences in applying mathematical ideas in the real world.

Zoltan Dienes and Principles of Variability

Dienes’ major message is also concerned with encouraging active student involvement in the learning process. Such involvement routinely employs a vast amount of concrete materials. He emphasized that learning mathematics should ultimately be integrated into one’s personality and thereby become a means of genuine personal fulfillment. His theory combines some ideas of Piaget and Bruner. He describes how mathematics is to be learnt and how it should be taught (both prescriptive and descriptive) insisting that, every learning should start with applications which the learner can actually experience and then progress to formal mathematical summary.

Dienes considers learning as a process of increasingly intricate play, which he classifies as primary and secondary. Primary play refers to any activity with materials and is aimed at gratifying immediate desires. In mathematics this involves the manipulation and investigation of materials for its own sake. Secondary play refers to any activity carried out with awareness and is aimed at reaching a goal beyond gratification of ideas. This involves building with the materials, discovering patterns and forming abstract conjectures.

Dienes was the architect of the '*multiple embodiment* or *multi embodiment principle*' which states that each mathematical concept should be taught using as many different examples and situations as possible. That is, perceptually different models should be used to teach one particular concept. The principle rests on the value of experiencing a mathematical concept in a variety of different settings. The more divergent the models are, the more likely it is for the learners to extract only the common characteristics and make abstractions. He designed the *Multi-base Arithmetic Blocks* (MABs), which is a set of structural apparatus meant to encourage learners to develop their mathematics through their own discoveries.



Multi-base Arithmetic Blocks

Dienes' theory of mathematics learning has four basic components or principles. These are the dynamic principle, perceptual variability principle, mathematical variability principle and the constructivity principle.

1) The *dynamic principle* suggests that true understanding of a new concept is an evolutionary process involving the learner in three temporally ordered stages. The first stage is the preliminary or play stage, where learner experiences the concept in a relatively unstructured but not random manner. Children exposed to a new type of manipulative material, naturally play with their newfound "toy". This is an informal activity but has to be provided by the classroom teacher followed by more appropriate structured activities at the second stage. The third stage is characterized by the emergence of the mathematical concept with ample provision for reapplication to the real world. This learning cycle should necessarily be completed before any new mathematical concept can become operational for the learner.

2) The *perceptual variability principle* suggests that conceptual learning is maximized when children are exposed to a concept through a variety of physical contexts or embodiments. The experiences provided should differ in outward appearance while retaining the same basic conceptual structure. When children are given opportunities to see a concept in different ways and under different conditions, they are more likely to perceive that concept irrespective of its concrete embodiment. When exposed to a number of seemingly different tasks that are identical in structure, children will tend to abstract the similar elements from their experiences. This process is known also as mathematical *abstraction*.

3) This *mathematical variability principle* suggests that the generalization of a mathematical concept is enhanced when variables irrelevant to that concept are systematically varied while keeping the relevant variables constant. For example, for the concept of *parallelogram,* this principle suggests that it is desirable to vary as many of the irrelevant attributes as possible. In this instance, the size of angles, the length of sides, and the position on the paper should be varied while keeping the only crucial attributes *a four-sided figure with opposite sides parallel* intact. Many persons erroneously believe that squares and rectangles are *not* parallelograms. This misconception has resulted because the appropriate mathematical variables (in this case angle size) had not been manipulated when they were taught the concept.

4) The *constructivity principle* states that “construction should always precede analysis.” It is analogous to the assertion that children should be allowed to develop their concepts in a global intuitive manner beginning with their own experiences. Dienes identifies two kinds of thinkers: the *constructive thinker* and the *analytical thinker*. He roughly equates the constructive thinker with Piaget's concrete operational stage and the analytical thinker with Piaget's formal operational stage of cognitive development. The constructive experiences should form the cornerstone on which all mathematics learning is based. At some future time, attention can be directed toward the analysis of what has been constructed; however, Dienes points out that it is not possible to analyze what is not yet there in some concrete form.

Implications of Dienes’ work

The principles lay emphasis on the importance of learning mathematics by means of direct interaction with the environment, requiring active type of physical and mental involvement on the part of the learner.

1) Individual and small-group activities would be used regularly because it is not likely that more than two to four children would be ready for the same experience at the same point in time. Thus whole-class (or large-group) lesson would be greatly deemphasized in order to accommodate individual differences in ability and interests.
2) The teacher’s role should include exposition as well as being a facilitator while students’ role should be expanded to assume a greater degree of responsibility for their own learning.
3) The newly defined learning environment would create new demands for additional sources of information and direction such as a learning laboratory containing materials such as computers and other modern technologies.

Key ideas

- Cognitive learning theory is concerned with both *how* children learn and with *what* it is that they learn.

- Major proponents of the cognitive learning theory include Jean Piaget, Jerome Brunner and Zoltan Dienes.
- Cognitive learning theory provides the major theoretical rationale for the promotion of active student involvement in the learning process.
- The Cognitive theory builds on two key underlying assumptions: (i) the memory system as an active organized processor of information and (ii) prior knowledge plays an important role in learning.
- Piaget believes that human intellectual development progresses chronologically through four sequential stages: sensory-motor, pre-operational, concrete operational, and formal operational.
- Key ideas and principles proposed by Piaget include assimilation, accommodation, and conservation.
- Key ideas and principles proposed by Dienes include the multi embodiment principle, multi-base arithmetic blocks.
- Bruner's instructional model is based on four key concepts: structure, readiness, intuition, and motivation.
- Bruner suggested three different levels of thinking namely enactive, iconic, and symbolic.

Reflections

- How have my exposure to the constructivist theory equipped me with the requisite knowledge and strategies to model a constructivist classroom that support students' mathematics learning?

Discussions

- Explain the major contributions made by Jean Piaget to the teaching and learning of mathematics.
- Give **three** implications of Bruner's theory to the mathematics teacher.
- Explain the *multi-embodiment principle* by Dienes.
- Explain the term reinforcement and the various forms of reinforcement.
- What does Bruner mean by the statement "*any subject can be taught effectively in some intellectually honest form to any child at any stage of development*"

SESSION 4: CONSTRUCTIVISM AND MATHEMATICS EDUCATION

In this session, attention is on constructivism and mathematics education.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain the constructivist view of learning;
ii. discuss the role of the teacher and the learner in the constructivist classroom;

Now read on …

Constructivism, an outgrowth of Cognitivist learning theory, views learning as an active process in which learners construct new concepts based upon their current or past knowledge. The learner is actively engaged in constructing both the knowledge acquired and the strategies used to acquire it. The learner selects and transforms information, constructs hypotheses, and makes decisions, relying on a cognitive structure to do so. Cognitive structure (i.e., schema) provides meaning and organization to experiences and allows the individual to "go beyond the information given". Constructivism emphasizes the importance of the active involvement of learners in constructing knowledge for themselves, and building new concepts based upon current knowledge and past experience. Three varieties of Constructivism are: Active learning, Discovery learning, and Knowledge building. All the three versions promote a student's free exploration within a given framework or structure. They demand that the teacher acts as a facilitator who encourages students to discover principles for themselves and to construct knowledge by working to solve realistic problems.

In a typical mathematics classroom manned by a constructivist teacher,

1. teaching strategies are tailored to student responses and students are encouraged to analyse, interpret, and predict information.

2. meanings of actions and words are negotiated by both teacher and students as they interact.

3. learners are encouraged to verbalize mathematical thinking, to explain and justify mathematical solutions and to learn to resolve complicating points of view. Constructivism in a way attempts to demystify mathematics and make it more accessible to all learners. Learners are exposed to posing and solving mathematical problems in social contents and discussing mathematics embedded in their own lives and environments.

4. appreciative counseling where learners are to be allowed to assess themselves and have regular dialogue with the teacher is encouraged. Learners feel free to comment on why they have not achieved a particular goal. Teachers regularly use alternative assessment procedures which encourage the use of a variety of instruments. Advocates of Constructivism suggest that *assessment becomes part of the learning process* so that students play a larger role in judging their own progress.

In conclusion, in a constructivist teaching and learning environment,
the **teacher** should:

1. orchestrate discussion among learners;

2. encourage learners to verbalize the mathematics they are constructing when doing activities;
3. encourage learners to make connections between different aspects of mathematics;
4. encourage learners to explain and justify their solutions;
5. encourage self and peer assessment;
6. make learning relevant to everyday life;
7. use a variety of resources to cater for different learning styles.

the **learners** on their part feel free to:

1. discuss work with peers and teacher;
2. pose and solve own problems; work in groups and generate mathematical problems.
3. verbalize the mathematics they are constructing when doing activities;
4. make connections between different aspects of mathematics;
5. explain and justify own solutions;
6. do self and peer assessment;
7. investigate own errors.

Key ideas

- Constructivism emphasizes the importance of the active involvement of learners in constructing knowledge for themselves, and building new ideas or concepts based upon current knowledge and past experience.
- The learning theories of John Dewey, Marie Montessori and David Kolb serve as the foundation of constructivist learning theory.
- Three varieties of constructivism are: Active learning, Discovery learning, and Knowledge building.

Reflections

- How has my knowledge of the constructivist theory help me to facilitate the teaching and learning of mathematics in the classroom?

Discussions

- Explain **four** things each that the teacher and the learner respectively should do in the constructivist teaching and learning environment.

UNIT 3: CONCEPT AND CONCEPT FORMATION

This unit introduces participants to the idea of concepts and concept formation in mathematics.

Learning outcome(s)
By the end of the unit, the participant will be able to:
1. explain the meaning of concept in mathematics;
2. explain factors that facilitate concept formation;
3. explain ways of communicating mathematics concepts to pupils.

SESSION 1: CONCEPTS IN MATHEMATICS

In this session, we will focus on the meaning of concepts and types of concepts in mathematics.

Learning outcomes

By the end of the session, the participant will be able to:
1. explain the meaning of a mathematical concept;
2. discuss the types of concepts in mathematics.

Now read on …

Definition of Concept in Mathematics

Skemp, defines a **concept** as 'the mental object, which results when we abstract from a number of examples or instances something which they all have in common. A concept is a mental representation of the common properties. It is some kind of lasting change, the result of abstracting which enables us to recognize new experiences as having the similarities of an already formed class'. Knowing what some of these mathematical objects are and how to identify them, equips us to easily study the relations among them and then to study how to use them. When we teach our students what some object is, how to identify it, we are teaching them the *concept* of that object.
Concepts are also seen as the ideas or abstractions formed as a result of categorizing data from a number of observations. They enable us to group together different things that have some similarities. They allow us to organize and store similar pieces of information efficiently. They help us to be more efficient in our communication and this tends to enhance comprehension and assist in transfer of learning.

Viewing mathematics as a hierarchy of concepts helps us to at least organize the mathematical knowledge that we wish to teach. Skemp views learning as a goal directed change of state of a **director system** which helps an organism to direct and organize its behavior towards a goal. The functions of a director system is governed by four emotions: **pleasure, confidence, displeasure and frustration**.

(i) *Pleasure* signals the approach of reaching a goal state;

(ii) *Confidence* signals the ability to reach a goal state;
(iii) *Displeasure* signals a retreat from a goal state; and
(iv) *Frustration* signals the inability to approach or reach the goal state.

The corresponding **anti-goal states** are also governed by the emotions of *fear*, a*nxiety, relief and security*. Emotions play a dominant part in the way we learn mathematics. The director system registers any mathematical problem that confronts us. It passes on the message to our emotional system which feeds back to the director system a message of *confidence* to spur us on to get on with the task, or a message of *anxiety* which discourages us to avoid the task.

Type of Concepts

Primary concepts are concepts that we learn through our experiences or interaction with the environment. They are derived directly from our sensory and motor experiences of the outside world. Examples of primary concepts are red, blue, green, or yellow; rectangle, triangle, or circle; heavy or light; sweet or sour; hot or cold; etc. The concept of *red* is formed when we see many red objects; the concept of *rectangle* by touching rectangular objects.

Secondary concepts are abstracted from primary concepts. Seeing sets of objects in singles, or in pairs, or in triples, etc, prepares the grounds for establishing the concepts of one, two, three, etc. Thus, in a sense, *one, two, three, four,* etc are secondary concepts formed from the ideas of oneness, twoness, threeness and fourness. Rectangle, triangle, or circle also combine to form the secondary concept of *shape*. Secondary concepts and higher order concepts thus depend on other concepts and so can only be formed if the learner has already formed these other concepts.

Some concepts are of **higher order** than others. Now when one, two, three, etc, are put together we form the higher order concept of *number* because that is the property shared by these concepts. Thus, if concept A is an example of concept B then B is said to be of a higher order than A. In addition if B is an example of C, then C is of higher order than both A and B. This idea of related concepts, of order between concepts and conceptual hierarchy, enables us to see clearly why mere definition is an inadequate mode of communication.

Key ideas

- A concept is a mental representation of the common properties abstracted from a number of objects.
- The three types of concepts are primary, secondary, and higher order concepts.
- **Primary concepts** are concepts that we learn through our experiences or interaction with the environment.
- **Secondary concepts** are abstracted from other concepts. They are built up by combining primary concepts.

- Higher order concepts are connected to all lower-level concepts in a complex hierarchy of abstractions

Reflections

- How has my exposure to the content of this session on concept prepared me to create learning environments which can offer students opportunities to form mathematical concepts?

Discussions

- Briefly distinguish between primary and secondary concepts, giving suitable examples.
- Explain the term *concept* in mathematics with two illustrative examples.

SESSION 2: CONCEPT FORMATION IN MATHEMATICS

In this session, we will limit our discussion to factors that facilitate concept formation in mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

i. discuss factors that facilitate concept formation;
ii. discuss ways pupils form concepts in mathematics.

Now read on …

Factors that Facilitate Concept Formation

The formation of concepts in mathematics are influenced by the following factors:

1. **Naming**

Names are often used to classify mathematical concepts. Recall that a concept is an idea. The name of a concept is a sound, or a mark on paper, associated with the concept. Names are closely linked with concepts and that makes some people to confuse concepts with name because for every example used, the name is also used. The use of a name in association with a concept helps us to classify it and to recognize it as belonging to an existing class. When we hear the same name repeatedly in connection with differing experiences our chance of abstracting their intrinsic similarities increases. It should now be clear that the meaning of a word is the concept associated with that word. A concept is **not** the word itself.

2. **Function or what the object does**

When two or more objects are similar what an object does becomes useful in forming its concept. Once the object is classified by what it does, we know how to behave in relation to it. We know that number can be classified as even, odd, prime, or composite, perfect square, perfect number, etc. A number is even because collection representing it can be paired, or a number is a perfect square because it is the product of a number by itself.

3. **Contrast**

A **non-example** refers to an object which does not belong to a collection of objects with similar characteristics. It is an odd object among a given collection. Providing contrasts or non-examples of a mathematics concept makes the concept more likely to be remembered and their similarities more likely to be abstracted across interval of space and time. The difference makes the similarity between them more noticeable. This fixes the borderline of a class. For example, in giving examples for rectangles we need to contrast them with 'non-rectangles' that is, other quadrilaterals like rhombus, kite, trapezium, etc.

4. **Noise**

Noise refers to any stimulus that is not very much wanted in a particular learning situation**.** When all the examples of a rectangle have the same colour, say blue, in teaching the concept of a rectangle, the blueness becomes a noise. Pupils can associate blueness with all rectangles and may in future find it difficult to classify a red rectangle under rectangles. This noise impedes the formation of a correct concept of a rectangle. If the interference by a noise is so much, the learning of that concept is impeded. A high noise occurs when the stimulus contains more of the unwanted things and this could impede the formation of a concept by an average learner.

Ways Pupils Form Concepts in Mathematics

Many children form some basic mathematical concepts even before coming to school to learn to add or subtract. Children form mathematical concepts through a sequence of processes of *discrimination*, looking for *similarities and differences*, *abstraction* and then *generalizing*.

1. **Discrimination**

When pupils manipulate materials in their immediate environment they usually find out the objects bearing common properties and put them into groups based on *classification* of their previous experiences and then fit of their present experiences into one of these classes. To classify is to collect together our experiences on the basis of observed similarities. This helps them to use past experiences to understand new ones**.** That is, when pupils interact with the materials they begin to **discriminate** and observe what belongs to where.

2. **Similarities and differences**

Pupils look for and describe the similarities and differences observed among the examples they confront. They use their varying past experiences to abstract certain invariant properties, which persist in their memory much longer.

3. **Abstraction**

To abstract means to extract what is common to a number of different situations and to disregard what is irrelevant. Abstraction is the process of formulating generalized concepts of common properties by disregarding the differences. Pupils use abstraction in concept formation when they begin to abstract the (main) invariant properties of the examples. For example, a pupil is able to learn the concept of “five“ by observing objects put in groups of five, e.g., 5 tables, 5 chairs, 5 sticks, etc. The ability to abstract the concept of five does not in any way depend on the type of objects involved in the process. What is invariant in all the examples is the “fiveness” (the size of the collection and not the materials involved).

4. **Generalization**

After the initial discrimination, classifying, and abstraction the child draws some general conclusions from a number of experiences, that is to say, they *generalize*. Pupils begin to make generalisations when they begin to address the question *Does this always work?*

Key ideas

- Factors that facilitate concept formation are naming, function or what the object does, contrast and noise.
- Children form mathematical concepts through a sequence of processes of discrimination, looking for similarities and differences, abstraction and then generalizing.
- Teachers should act as facilitators in the learning process for proper formation of concepts by pupils

Reflections

- What are some of the experiences I went through in concept formation at the basic/secondary/tertiary level(s)? How have these experiences prepared me to facilitate students proper formation of concepts?

Discussions

- With illustrative example in each case, explain **three** factors that influence concept formation in mathematics.
- Discuss four ways that children can form mathematical concepts.

SESSION 3: COMMUNICATING MATHEMATICS CONCEPTS

This session focuses on how to communicate mathematical concepts to learners and the principles that guide the teaching of mathematical concepts.

Learning outcomes

By the end of the session, the participant will be able to:

1. communicate mathematics concepts to learners;
2. explain the principles that guide the teaching of mathematical concepts.

Now read on …

Moves in Communicating Mathematics Concepts to Learners

Teacher **actions** or **moves** in communicating mathematics concepts to students include using suitable examples and non-examples, naming, and finally using definition.

1. By examples

Identifying and providing suitable *examples* of the concepts to be formed **with** and later **without** reasons is a common move used by teachers to communicate concepts to learners. The teacher is to provide the materials from the environment for learners to experience and interact with and to ensure that the stimuli provided are relevant and embody the concepts to be taught. For example, in teaching the concept of rectangle, you need to provide several examples of rectangular cutouts and objects that are rectangular in shape.

Learners on their part are to:

a. identify the examples of the concepts;
b. abstract the common properties from the examples;
c. use properties to classify objects belonging to the concepts. E.g. a rectangle has only 4 sides. It has 4 right angles and the opposite sides are equal.

2. By Naming

Teacher now provides an appropriate **name** associated with the examples of the concept saying for example *This is a rectangle, this is also a rectangle. All these are examples of a rectangle*.

3. By Non-examples

Later teacher provides **non-examples** of the concepts first **with** and later **without** reasons for pupils to contrast and identify the boundaries. Suitable non-examples to use in teaching the concept of rectangles are trapezium, a rhombus, a kite, etc. The students learn to use properties to exclude objects not belonging to the concepts.

4. **By Definition**

It is quite difficult for both teacher and students to communicate concepts by definition. Providing a **definition** of the concept depends on the level and the cognitive development of the pupils. Definitions help to identify the properties associated with the concept and enable us to retrace our steps, that is, to identify where a concept starts and ends. Definitions show the relationship of one concept to another in its hierarchical form. Definitions add precision to the boundaries of a concept, once formed, *E.g. A rectangle is a four-sided plane figure (a quadrilateral) with four equal angles and equal opposite sides.*

Principles that Guide the Teaching of Mathematical Concepts

There are two principles suggested by Skemp to guide teachers in teaching concepts in mathematics.

Principle 1: Concepts higher in order of hierarchy than those a learner already has should not be taught by definition. Teachers must arrange suitable examples for the learner to experience. It is not appropriate to communicate by definition concepts that are of a higher order than those, which a person already has. Teachers have to arrange for the learner to encounter suitable examples that contribute to the formation of the new concepts.

Principle 2: Because these mathematical examples are invariably concepts themselves the teacher must ensure that learners possess these concepts before using them in definitions. Definitions in mathematics must be made to evolve naturally from previous knowledge, models or real experiences that a pupil can relate to. Mathematical definitions are generally very concise, contain technical terms, and require an immediate synthesis of the formation if our learners are to understand, but most learners cannot operate on the abstract or formal level until high school.

Useful Guides to the Teacher in Selecting Teaching Actions.

1. Examples that the teacher uses must be concepts the learners already know. This is because pupils understand a concept when they can relate it to other concepts they already know. The learning of a higher order concept can take place only after the lower order contributory concepts have been formed. In order to rely on pupils' relevant previous knowledge, the contributory concepts must be formed before new ones are learnt.

2. When pupils are involved in practical activities and eventually discover things for themselves they learn mathematics better. Pupils are personally involved (they **DO** mathematics). All pupils need some form of concrete experiences. Older pupils may need fewer than younger ones and pupils of higher ability may need less than those of low ability but all pupils should have some.

3. As much as possible, arrange adequate and diverse experiences for learners to go through. The appropriate experiences must possess common properties to be abstracted that will form the concept. Two examples must possess common properties to be abstracted (i.e. common properties must be similar but other features not wanted must differ). The use of examples and non-examples helps students to grasp the concepts very well.

4. Create a platform to encourage students to discuss problems among themselves and with the teacher. This is because mathematical thinking is an essential part of problem solving and pupils would think for themselves when they are given the chance to ask questions. Students should **TALK** about the mathematics they are learning.

5. Teachers must use the correct communication moves such as practical discoveries, use of suitable teaching-learning materials and examples to support pupils form concepts. Methods of instruction should accommodate the natural thought processes of the pupil. Teachers should select appropriate experiences that will challenge pupils' level of thought. Ineffective work on the part of the teacher leads to anxiety or the fear of mathematics by learners. If at one level an abstraction is imperfectly done, there is the danger of learners misunderstanding the rest of the further concepts that use the first as a contributory one.

6. Teachers must be inventive and use appropriate examples and further examples. They must use other contributory concepts to explain new concepts. They can succeed if they have a clear understanding of the concepts themselves so that they can easily analyse the various contributory concepts on which to build the new concepts.

7. Make the learning of mathematics devoid of monotony and intimidation and most importantly make it more meaningful to the pupils because they learn better in a dynamic and enjoyable atmosphere. The attitude of the teacher reflects the interest and desire of the pupils to work.

8. Use concrete learning experiences to bring meaning to the symbolic representations necessary to embody mathematical concepts. The desired learning sequence should be from concrete, manipulative experiences to semi-concrete (graphical and pictorial experiences) and finally to abstract symbolic experiences. In every mathematics lesson, children should be given the opportunity to **DO**, **TALK** about and then **RECORD** their work.

9. Teacher should talk about the usefulness of mathematics in both careers and in everyday life.

   In general, pupils learn best:
   1) when they know their teachers believe in them;
   2) when they are loved and praised;
   3) when they are ready;
   4) by doing;
   5) when what they are learning is related to what they already know;
   6) step-by-step, going from concrete to abstract;
   7) when they use what they have learned more often;
   8) from each other, as well as from adults;
   9) when they experience success in learning. They are encouraged to learn more;
   10) when they receive immediate feedback to know whether or not they have learned.

Key ideas

- Teacher **actions** or **moves** in communicating mathematics concepts to students include using suitable examples and non-examples, naming, and finally using definition.

- There are several useful guides that teachers can employ to select teaching actions. Fundamentally, teaching examples that the teacher uses must be concepts the learners already know.
- Skemp suggested two principles to guide the teaching of mathematical concepts.

Reflections

- How has the content of this session imparted my teaching of mathematical concepts to students at the JHS level?

Discussions

- Identify eight ways pupils learn best.
- Explain the moves the mathematics teacher employs in communicating concepts to students.
- Explain eight useful guides that the mathematics teacher can employ to select his/her teaching actions.

UNIT 4: SCHEMA AND UNDERSTANDING IN MATHEMATICS

This unit introduces participants to the idea and functions of schema in mathematics.

Learning outcome(s)

By the end of the unit, the participant will be able to:
1. explain the idea of a schema in mathematics;
2. explain the functions of schema in mathematics.

SESSION 1: FUNCTIONS OF A SCHEMA IN MATHEMATICS

This session focuses on the idea of a schema and understanding in mathematics. It also discusses the functions of schema.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain the idea of a schema in mathematics;
ii. discuss the role of schema in understanding mathematical concepts.

Now read on …

Meaning of Schema in Mathematics

A schema is a mental or conceptual structure formed from the experiences that an individual has and stores in the mind on which he/she builds. David Tall defines a schema as a coherent mental activity in the mind of an individual which changes with time. Schemas can develop and become more versatile; they can decay; they can change by conscious and unconscious reformulation of ideas as the child attempts to make a coherent pattern out of the universe he/she lives in. Piaget also refers to a schema as the cognitive or mental structure by which individuals intellectually adapt to and organize the environment. Schemas adapt and change with mental development. They never stop changing or becoming more refined.

For example, a child adding fractions thus $\frac{3}{8}+\frac{5}{8}=\frac{8}{16}$. The child has a schema for adding natural numbers but such existing schema did not permit him or her to perceive the differences.

As a child grows, the schemata become more differentiated, less sensory and more numerous; and the network they form becomes increasingly more complex. A child for instance needs additional concept of place value in order to add 37 + 45 correctly, otherwise the child could easily give the

answer as $\begin{array}{r} 3 \quad 7 \\ + \;\; 4 \quad 5 \\ \hline 7 \;\; 12 \\ \hline\hline \end{array}$.

The child's original schema on addition has been expanded and so the need for additional information before child can cope with the new situation.

Functions of a Schema

Three main related functions of a schema have been identified.

1. **A schema is integrative in nature:** This concerns the interrelatedness of mathematical concepts. Schemas help us in identifying one object as belonging to a class of other objects. When faced with a mathematical problem we begin to bring into focus all past experiences that can be integrated to solve that problem. Thus, the more other schemata we have available, the better our chance of coping with the unexpected. Note that mathematical concepts are hierarchical in nature. The concept of rectangles is embedded in the concept of parallelogram and that of parallelogram is embedded in the concept of quadrilaterals. A schema thus helps to show the interconnections among all the related concepts.

2. **A schema can be used as a tool for further learning:** Most new learning depends on some fundamental previous ones. Familiar concepts can be used to acquire new knowledge. An existing schema becomes an indispensable tool for the acquisition of further learning. For example, knowledge of $5+3 = 8$ aids the understanding of $5y + 3y = 8y$ and $5\sqrt{2} + 3\sqrt{3} = 8\sqrt{5}$. That is, prerequisite schemas, which we build up in the course of our early learning, make it easy or difficult for us to understand later topics. For example, knowledge of properties of a rectangle and area of a rectangle aids the learning of area of a triangle and area of a parallelogram.

3. **Schemas make understanding possible:** To understand something means to assimilate it into an appropriate schema. Understanding is the forming of the right connection between the new material and the existing schema. When there is a better internal organization of a schema understanding is improved and there is no stage at which this process is complete. One obstacle to further increase in understanding is the belief that one already understands fully. Note that when appropriate schemas are not learnt well they become counterproductive. Some may yield correct or desired results at the initial stages but may not be helpful in the future applications. When we understand something well we tend to behave appropriately in a new class of situations.

Key ideas

- A schema is a mental or conceptual structure formed from the experiences that an individual has and stores in the mind on which he/she builds.
- Schemata are intellectual structures that organize events as they are perceived by the organism into groups according to common characteristics.
- Three main functions of a schema identified are that it is integrative, used as a tool for further learning and for making understanding possible.

Reflections

- What are some of my experiences in terms of building schemas and network of connections to facilitate understanding of mathematical concepts?

Discussions

- With one illustrative example, explain what is meant be *mathematical schema*.
- With one illustrative example in each case, explain the main functions of a schema in mathematics.

SESSION 2: FORMS OF UNDERSTANDING

In this session, we will discuss two forms of understanding suggested by Skemp. These are *relational* and *instrumental* understanding.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain the forms of understanding in mathematics;
ii. discuss the advantages and disadvantages of instrumental understanding;
iii. discuss the advantages and disadvantages of relational understanding;

Now read on …

Relational Understanding: This corresponds to intelligent learning Skemp refers to as understanding itself. The learner knows *both* '*what to do and why*'. Learner can apply the rules and can explain why the formula he or she is using works. The learner knows which formula to use, when to use it and why use it. David Tall refers to it as 'understanding which comes about through the search for coherence'. This understanding occurs when connections between concepts take place. Teachers would normally want the learners to demonstrate that they can perform mathematical skill and at the same time to understand what they are doing.

Instrumental Understanding: Instrumental understanding corresponds to habit learning, often referred to as "*rules without reasons*". Instrumental understanding is simply an exercise of memory and no more. Learner knows how to apply the techniques and methods or rules to solve mathematical problems but does not understand why the rules being used work. For example, a teacher reminds the class that the area of a rectangle is given by $A = l \times w$, where A is the area, l is the length and w is the width of the rectangle. A pupil says he does not understand and the teacher gives a further explanation as follows: "The formula says that to find the area of a rectangle, just multiply the length by the width." The pupil says "okay", I understand and uses the formula to work out the given exercise and gets all the answers correct. The pupil shows understanding, but this is an instrumental understanding.

Instrumental understanding occurs when we teach our pupils to use "invert and multiply" in division by a fraction; and in multiplying a fraction by a fraction, i.e., $\frac{5}{9} \times \frac{2}{7} = \frac{5 \times 2}{9 \times 7} = \frac{10}{63}$, "multiply the numerators to make the new numerator and multiply the denominators to make the new denominator".

Advantages of Instrumental Understanding

1. **Instrumental learning is usually easier to understand within its own context**

   Topics, such as product of two negative numbers, or dividing by a fractional number, are quite difficult to understand relationally. Rules such as '*Minus times minus equals plus'*, are quite easy to remember and can provide a page of right answers more quickly and easily.

2. **The rewards are more immediate and more apparent**

   Pupils getting right answers means success and this makes them happy. Success restores their self-confidence, and instrumental learning can arguably help them achieve this more quickly.

3. **One can often get the right answer more quickly and reliably**

   Less knowledge is usually involved, and so it is often very easy to get the right answer more quickly and reliably by instrumental thinking than relational thinking.

Advantages of Relational Understanding

1. **It is more adaptable to new tasks:** Because a pupil has attained a relational understanding, he or she knows what particular method works and also why it works and this enables him or her to relate the approach to new problems. When an idea is fully understood in mathematics it is more easily extended to learn a new idea. The learning of new concepts and procedures becomes easy.

2. **It is easier to remember:** Mathematics learnt relationally is easier to remember because the understanding of the interconnections between concepts allows them to be remembered as part of a connected whole instead of a list of separate rules. Relational understanding consists partly in seeing all of the relationship in the rules for area of triangles, rectangles, parallelograms,

trapeziums. Knowing how they are inter-related enables one to remember them as parts of a connected whole, which is easier. For example, knowledge of equivalent fractions ties together rules concerning common denominators, reducing fractions, and changing between mixed numbers and whole numbers.

3. It enhances memory: Memory is a process of retrieving information. There is much less chance that information learnt relationally will deteriorate. The connected information is more likely than disconnected information to be retained over time and retrieval is also easier. Connected information provides an entire web of ideas to reach for. If what you need to recall seems distant, reflecting on ideas that are related can usually lead you to the desired idea eventually.

4. It can be effective as a goal in itself: Relational understanding is intrinsically rewarding especially when new information connects ideas already possessed. The new knowledge fits and makes sense. External rewards and punishments are greatly reduced and the 'motivational' side of a teacher's job is made much easier. Children who learn by rote are often motivated by external means: for the sake of a test, to please a parent, from fear of failure, or to receive some reward. Such rewards may be effective in the short run but do very little to encourage a love of a subject when the rewards are removed.

5. It is organic in quality: The satisfaction people get from relational understanding makes them try to understand any new material relationally, and also to actively seek out new material and explore new areas. Relational understanding is self-generative. New understandings are generated and this suggests a kind of snowball effect. When knowledge gained is found to be pleasurable, people are likely to seek or invent new ideas on their own, especially when confronting problematic situations.

6. It improves problem-solving abilities: The solution of novel problem requires transferring ideas learned in one context to new situations. When concepts are embedded in a rich network, transferability is significantly enhanced. Learners are well equipped to apply knowledge gained to solving real life problems.

7. It improves attitudes and beliefs: Relational understanding also has an affective effect. The learner tends to develop a positive self –concept about his or her ability to learn and understand mathematics. There is a definite sense of "I can do this! I understand!" Mathematics then makes sense to them. Learners tend to have a positive view about mathematics itself.

Disadvantages of Relational Learning

Situational factors which contribute to the difficulty in the use of relational teaching include:

1. **Over-burdened syllabi:** Many teachers complain about the number of topics to cover in the mathematics syllabus. There is a high concentration of the information in the content of mathematics because mathematical statements are largely condensed into single lines. Teachers believe that coverage of the syllabus can only be achieved by instrumental teaching but not relational teaching.

2. **The backwash effect of examinations:** Success in examinations for future employment has become the main concern of many schools. Students' main goal is to answer correctly a sufficient number of questions. Both teacher and students would like to attain this success and instrumental learning and teaching is more likely preferred.

3. **Psychological difficulty for teachers to accommodate:** It is psychologically difficult for some teachers to accommodate (re-structure) their existing and long-standing schemas.

4. **Difficulty of assessment of whether a person understands relationally or instrumentally** It is very hard to make valid inference about the mental processes by which a pupil arrives at a solution. In a teaching situation, talking with the pupil is almost certainly the best way to find out; but in a class of thirty or more, it may difficult to find the time.

An individual teacher might choose to teach for instrumental understanding on one or more of the following grounds:
1) That relational understanding normally takes too long to achieve.
2) That relational understanding of a particular topic is too difficult.
3) That a skill is needed for use in another subject (e.g. science) before it can be understood relationally with schemas currently available to the pupils.
4) That he is a junior teacher in a school where all other mathematics teaching is instrumental. That is, being influenced by current practices by the experienced teachers.

Key ideas
- Skemp suggested two forms of understanding, *relational* and *instrumental*.
- Relational understanding corresponds to intelligent learning also referred to as understanding itself.
- Instrumental understanding corresponds to habit learning often referred as "rules without reasons."
- Both relational and instrumental understanding have advantages and disadvantages.

Reflections
- How has my experiences with both relational and instrumental understandings influenced my teaching and learning of mathematical concepts?

Discussions

- Distinguish between *instrumental* and *relational* understanding in mathematics learning with **one** illustrative example in each case.
- Give **two** advantages and two disadvantages each of *instrumental* and *relational* understanding in mathematics.

SESSION 3: SCHEMATIC LEARNING

This session focuses on schematics learning and its associate concepts of assimilation and accommodation. Also, the session discusses the implications of schemas for learning mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain schematic learning;
ii. distinguish between assimilation and accommodation;
iii. discuss the implications of schemas for learning mathematics.

Now read on …

Schematic learning, also often referred to as intelligent leaning is learning which uses existing schemas as tools for the acquisition of new knowledge. It is learning that relies heavily on prerequisite knowledge or relevant previous knowledge of learners where related ideas are activated. For example,

1) The division task $\frac{1}{2} \div 3 = ?$ can be schematically learnt when a whole object or sheet of paper is divided into two equal parts, and one half is again divided into three equal parts, then we find out what fraction each small division makes with the whole.
2) The formula for surface area of a closed cylinder = $2\pi r^2 h + 2\pi r^2$ is schematically obtained through practical discovery activity using the net of a cylinder and knowledge of areas of rectangle and circle as pre-requisites.
3) Using the prerequisite knowledge of adding objects like ‘ 2 mangoes plus 3 mangoes gives 5 mangoes’ to explain the addition of like fractions $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$, by reading it simply as ‘two-sevenths plus three-sevenths equals five-sevenths’ that is, adding the objects called “sevenths”.

Schemas are important for intelligent learning or long-term learning because:

a) they make understanding possible, and thereby adaptable to new situations;
b) they provide a rich source of plans of action and techniques for a wide variety of application;

c) shared schemas have important social functions because, they facilitate co-operation on the basis of shared understanding, and plans which fit together to achieve shared or compatible goals.

In the long run:

d) when schematic learning takes place, it is much easier;
e) schematically learnt materials are better retained;
f) future learning is also easier: when learning takes place with understanding, new concepts are formed and connected with an appropriate schema.

All learning must be related to the existing knowledge, concepts and structures because schematic learning has a triple advantage over rote learning.

1. Schematic learning leads to learning more efficiently what we are currently engaged in.
2. It helps in preparing a mental tool for applying the same approach to future learning tasks in that field.
3. In using this tool we consolidate the earlier content of a schema.

Schematic learning may take longer time. It has selective effect on our experiences and this may hinder or facilitate future learning. When new learning fits into an existing schema they are better remembered. Those, which do not fit into it, are largely not learnt at all, and what is learnt is soon forgotten. Unsuitable schemas become a handicap in our future learning. Schemas may fail to undergo change and thereby become an obstacle to new learning. Teachers should lay appropriate foundation for leaners to build on. The teacher must select contributory schemas that would enhance understanding of the concept to be learnt or already learnt.

Two **implications** of schematic learning for teaching are as follows:

1. Ensure that the new concepts embodied in the learning materials we provide are such as can be assimilated to their existing schemas so that children can learn with understanding. Base teaching on pupils' existing knowledge (RPK). There is need for some diagnostic assessment to establish pupils' prerequisite knowledge.

2. Give particularly careful thought to the foundation schema in every topic because:
   a) schemas are highly selective and what is well assimilated is remembered more easily than what cannot. Ideas which do not fit our existing schemas are likely to be ignored, rejected or forgotten. Teachers should select the right schemas to start with pupils.
   b) we at times encounter new ideas which do not fit our existing schemas, but which we cannot ignore, or reject. It is not enough just to expand our present schemas but to make radical changes- to replace some of the existing concepts by different ones, to make different connections. This is a process called '**reconstruction**" also referred to as **accommodation**, which is usually "unwelcome". Schemas are important parts of our mental equipment and so we find it threatening when we have to reconstruct them.

Negative numbers were initially referred to as “absurd”, “fictitious” and “unnatural monstrosities”. These are natural defensive reactions and so must be expected from our students. We can minimize this among our students if we carefully select the foundation schemas.

Assimilation and Accommodation

Two forms by which schemas adapt to new situations are *assimilation* and *accommodation*. **Assimilation** is the cognitive process by which a person integrates new perceptual, motor or conceptual matter into existing schemata or patterns of behaviour- i. e., comprehending a new experience into existing ones. This process allows for the growth of schemata. Assimilation is ‘comfortable adaptation’. It gives the feeling of mastery and is usually enjoyed. For example, 9 oranges plus 8 oranges give 17 oranges is easily extended to $\mathbf{9\sqrt{2} + 8\sqrt{2} = 17\sqrt{2}}$ by process of assimilation. The new schema is easily absorbed into the existing schema.

Accommodation is a correcting process by which we either restrict or broaden our schemas. In accommodation, the individual modifies those internal cognitive structures to conform to the new information and meet the demands of the environment. When confronted with a new stimulus, a child tries to assimilate it into existing schema. Sometimes a stimulus cannot be placed or assimilated into a schema because there are no schemata into which it readily fits. A modification may have to be done to an existing one in order to accept the new one. The existing schema has to *undergo a structural change* before absorbing the new knowledge. The original schema is not overthrown, but it becomes part of the new one. An accommodation takes place when we move from operations on whole numbers to operations on fractional numbers which involves new characteristics (equivalent fractions), etc. For instance, children who have learnt that the symbol 3 represents *three* have to expand their concept when confronted with the numeral 36. The symbol 3 now represents *thirty*. Accommodation is ‘uncomfortable adaptation’ – one is ‘forced’ to modify existing schema to fit the new one. For that matter accommodation sets up imbalance between our existing concepts and our new ones. As we study mathematics, we extend existing schemas through assimilation and reconstruct temporary schemas through accommodation.

Implications of Schemas for Learning Mathematics

The following are the implications of schemas for learning mathematics.

1. Early schemas learnt inappropriately may make assimilation of later ideas much more difficult or impossible. Learners who are deficient in so many schemas may not achieve any level of understanding of higher level mathematics.

2. Successfully manipulating symbols and rules does not necessarily mean formation of appropriate schemas. Some learners tend to have bits and pieces of concepts memorized and do not have relationship existing between schemas.

3. When many rules are accumulated children may not be able to memorize them. Memorization hampers retention and understanding especially when the learning materials increase. Use appropriate schemas through mathematically related situations because appropriate schemas help long term learning e.g. using the ‘balance scale’ to teach solving simple equations. If we add (or subtract) the same weights to each pair of the scale, balance is preserved. This model justifies the rule ‘take a number to the other side and change the sign’.

4. Make sure that schematic learning (not rote learning) takes place. Do not rush learners to get to the final solution or rule rather go at a slower pace when accommodation is to take place and endeavor to check progress carefully.

5. Ask learners to explain concepts in their own words. This helps them to organize their schemas. Or it may help the teacher with insight to improperly developed schemas. Provide contextual application problems to help prepare the learner for transfer of concepts.

6. Plan on long-term basis by:
   a) laying well-structured foundations of basic mathematical ideas on which the learner can build future learning i.e. teach learners mathematics.
   b) helping learners to learn mathematics- learners to look for their own patterns and discoveries. This increases the power of the learner’s ability to learn.
   c) assisting learners to accommodate schemas i.e. replace existing schemas with better ones. This is difficult because children feel secured with the old schema and cannot leave it until such time that they feel comfortable with the new schema and have seen that it produces a better result. Teachers must therefore ensure that they explain mathematical ideas to the understanding of the children to enable them see the need to give up old schemas.

Key ideas

- Schemas adapt to new situations through *assimilation* and *accommodation*
- In assimilation, the individual absorbs new information, fitting features of the environment into internal cognitive structures.
- In accommodation, the individual modifies those internal cognitive structures to conform to the new information and meet the demands of the environment.
- Schematic learning has a triple advantage over rote learning.

Reflections

- What are some of the experiences in terms of assimilating and accommodating mathematical ideas or concepts? How have these experiences affected my teaching of mathematical concepts to students at the JHS level?

Discussions

- What is meant by schematic learning? What is the triple advantage schematic learning has over rote form of learning?
- Explain the ***two*** major implications of mathematical schemas to the mathematics teacher in teaching mathematics.
- With ***two*** mathematical illustrations in each case, distinguish between *assimilation* and *accommodation* in mathematics.

UNIT 5: INTUITIVE AND REFLECTIVE THINKING AND DISCOVERY LEARNING

This unit introduces participants to intuitive and reflective thinking in mathematics. The unit also discusses the discovery learning in mathematics.

Learning outcome(s)
By the end of the unit, the participant will be able to:
1. explain intuitive thinking;
2. explain reflective thinking;
3. explain discovery learning in Mathematics.

SESSION 1: INTUITIVE THINKING

In this session, we will concentrate on intuitive thinking and its characteristics.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain intuitive thinking in mathematics
ii. discuss the characteristics of intuition

Now read on …

Definition of Intuition in Mathematics

Intuition is ‘knowing something without knowing exactly how you know it. It involves accepting the truth of a statement without first reflecting on it. Intuition consists of the immediate apprehension, without the intervention of any reasoning process, of knowledge or mental perception (Oxford English Dictionary). It is a sudden realized perception or knowledge in mathematics, which cannot be explained because we have not subjected it to logical analysis to confirm the intuitive acceptance.

When confronted with the task: *Continue the patterns: 1, 2, 4, . . .* , two quick possible responses are: 1, 2, 4, 8, 16, 32, 64, 128 . . . and 1, 2, 4, 7, 11, 16, 22, 29, . .
By impulse the explanation could be *I really had no idea. I suppose I just saw the numbers doubling. Or, I suppose I just saw the numbers increasing by 1, 2, 3, 4, etc.*
Similarly, the first impulsive response to the question “How many squares are there in a 5 by 5 square grid?” is 25. The quick explanation may be just *squaring 5 or just counting*.
The impulsive responses to each of the problems above is an *intuition* which comes with no `consciousness of the intervention of any reasoning process.

Characteristics of Intuition

1. Intuition comes as a sudden flash- a 'Eureka' which does not seem to follow any logical sequence of steps.
2. It seems to be built upon a broad understanding of the field of knowledge involved.
3. The individual using this process seems to be characterised by bold guessing.
4. The intuitive thinker arrives at an answer with very little awareness of how s/he has got it. There is no immediate conscious awareness of where the answer had come from. Intuition does not follow a specific algorithm, or specific process to arrive at solution. It is a mere conjecture, a hit or miss affair and does not seem to follow order.
5. Intuition is difficult to describe because what goes on in intuition is a multiple criteria. There may be too many intervening variables. It does not proceed step by step. The thinker cannot report adequately each step to another person. The answer may be right or wrong.
6. Interpretation and judgement may not be straightforward and so it has to be proved. The intuitive thinker is often unable to give an explanation of why he or she had arrived at the answer and may even end up quite puzzled.
7. Because intuition is non-algorithmic, it is often uncertain. One is not sure what one feels is right or wrong.
8. There is self-regulation in intuitive thinking. Our thinking does not come from outside. What we perceive comes from our minds and we are in control of our minds. Teachers are encouraged to build on the learner's intuition.
9. It usually comes after the problem has been put aside, often when least expected. Thus an incubation period is necessary.
10. Because intuition comes suddenly and effortlessly, it must be put into use, otherwise it seizes to be an intuition. Fermat conjectured that $2^{2^n}+1$ is always a prime number. This was used to generate the prime numbers 5, 17, 257, and 65537. It remained a conjecture for some 100 years before it was detected that the fifth number generated, 4,294,967,297, is not a prime, it is divisible by 641.

Key ideas

- Intuition is a sudden realized perception or knowledge in mathematics, which cannot be explained because we have not subjected it to logical analysis to confirm the intuitive acceptance.
- There are several characteristics of intuition. Basically, intuition comes as a sudden flash- a 'Eureka' which does not seem to follow any logical sequence of steps.

Reflections

- How has my reading of this session equipped with the relevant knowledge and strategies to handle intuitive thinking?

Discussions

- Explain the term *intuition* in mathematics.
- Explain six characteristics of intuitive thinking.
- Outline two ways in which intuitive thinking can facilitate concept formation in mathematics.

SESSION 2: REFLECTIVE THINKING

In this session, we will confine our discussion to reflective thinking in mathematics and its role of the teacher in promoting reflective thinking.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain reflective thinking in mathematics;
ii. explain the role of the teacher in promoting reflective thinking;
iii. discuss the implications reflective thinking for teaching and learning.

Now read on …

Reflective Thinking

Reflective thinking is a part of the critical thinking process referring specifically to the processes of analysing and making judgments about what has happened. Here, learners are aware of and control their learning by actively participating in reflective thinking – assessing what they know, what they need to know, and how they bridge that gap – during learning situations. The ability to think reflectively is one's ability to understand the thought process by looking back at what has been done to find a solution to a problem.

Critical thinking refers to thinking that is purposeful, reasoned and goal directed - the kind of thinking involved in solving problems, formulating inferences, calculating likelihoods, and making decisions. It involves "the use of those cognitive skills or strategies that increase the probability of a desirable outcome. Critical thinking is sometimes called *directed thinking* because it focuses on a desired outcome. Note that *critical thinking* involves a wide range of thinking skills leading toward desirable outcomes and *reflective thinking* focuses on the process of making judgments about what has happened.

Reflective Intelligence comes about when we subject the intuitive knowledge to logical analysis later to confirm what we first accepted intuitively. We make use of both intuitive and reflective modes of intelligence. We accept facts first and think about them later before we fully accept them. This happens when we try to explain how and why we do certain things. Reflections help in correcting schemas and lead to the discovery of new ones. Communication also helps to link ideas with symbols and makes us interact our ideas with those of other people. Trying to teach a topic helps us to clarify our own thinking about the topic. You understand the topic the more if you attempt to teach somebody. It gives better results than practising on your own. As mathematics teachers, we should encourage learners to explain their solutions and also form and lead study groups.

Reflective thinking provides learners with the skills to mentally process learning experiences, identify what they learned, modify their understanding based on new information and experiences, and transfer their learning to other situations. It is significant to prompt reflective thinking during learning to help learners develop strategies to apply new knowledge to the complex situations in their day-to-day activities.

Role of the Teacher in Promoting Reflective Thinking

To promote reflective thinking skills in students, the teacher has to:

1. provide enough wait-time for students to reflect when responding to inquiries.
2. provide emotionally supportive environments in the classroom encouraging re-evaluation of conclusions.
3. prompt reviews of the learning situation, what is known, what is not yet known, and what has been learned.
4. offer reliable tasks involving ill-structured data to encourage reflective thinking during learning activities.
5. provide a less-structured learning environment that prompts students to explore what they think is important.
6. prompt students' reflection by asking questions that seek reasons and evidence.
7. provide some clarifications to guide students' thought processes during explorations.
8. provide social-learning environments such as those inherent in peer-group works and small group activities to allow students to see other points of view.
9. provide reflective journal to write down students' positions, give reasons to support what they think, show awareness of opposing positions and the weaknesses of their own positions.

Implications for Teaching and Learning

The implications of intuitive and reflective thinking processes for the mathematics teacher include:

1. Learning at the intuitive stage depends on the way materials are presented to the children. Concepts too far removed from the child's existing schemas may not be easily assimilated. The teacher of mathematics must fit the mathematical materials to the state of the development of the learner's mathematical schemas.

2. Teacher must fit his or her manner of presentation to the modes of thinking of which learners are capable. Teacher must encourage learners to ask questions and teachers must give explanations [build on pupil's intuition]

3. Teacher must gradually reduce the learner's dependence on him or her – learners must be encouraged to explore and work on their own. Be gradually increasing learners' analytic abilities to the stage at which they no longer depend on the teacher.

4. Be a life-long learner. Teachers must remain active learners and learners must be encouraged to guess in class. Intuitive and analytic thinking seem to be complementary. The intuitive thinker may often arrive at solutions, which he would not achieve at all working analytically. However, there is need to check his solutions analytically.

Key Ideas

- Reflective thinking is a part of the critical thinking process referring specifically to the processes of analysing and making judgments about what has happened.
- Reflective thinking is an active, persistent, and careful consideration of a belief or supposed form of knowledge, of the grounds that support that knowledge and the further conclusions to which that knowledge leads.
- Critical thinking refers to thinking that is purposeful, reasoned and goal directed - the kind of thinking involved in solving problems, formulating inferences, calculating likelihoods, and making decisions.
- *Critical thinking* involves a wide range of thinking skills leading toward desirable outcomes.
- *Reflective thinking* focuses on the process of making judgments about what has happened.

Reflection

- How has this session equipped me with the relevant knowledge to critically analyze and apply both critical and reflective thinking processes in the classroom?

Conclusion

- Mention ***six*** ways in which the mathematics teacher could help students develop reflective thinking during mathematics lesson.
- Distinguish between *reflective thinking and critical thinking*?
- Explain five ways the teacher can promote reflective thinking among students
- Explain three implications each of intuitive and reflective thinking to the mathematics teacher.

SESSION 3: DISCOVERY LEARNING IN MATHEMATICS

In this section, we will restrict our discussion to discovery learning in mathematics. The strengths and weaknesses of discovery learning will also be discussed.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain discovery learning in mathematics;
ii. discuss the strengths and weakness of discovery learning.

Now read on …

Discovery Learning in Mathematics

Teaching is mainly to facilitate or simplify learning. Learning is coming to know something one did not already know. Two types of **knowing** have been identified as "*knowing how*" and "*knowing that*". "Knowing that" deals with factual knowledge. One can know *that* $3 \times 4 = 12$, or $18 \div 3 = 6$. "Knowing how" is the ability to do something. One can know *how* to find lowest common multiple, or highest common factor of two natural numbers. One learns *how* to do something by trying to do it, failing, and trying again. Learning *how* is a process of trial and error.

"Teaching that" consists of providing students with data and checking their retention of it. Teaching someone how to do something, however, entails observing his doing of it, correcting his mistakes, and suggesting how the mistakes might be avoided. The most significant part of education at all levels is learning *how*, not learning *that*. It is not beneficial to know *that* without knowing *how* one knows that (which requires being able to say why anyone should believe it to be true).

Discovery learning is learning in which the learner finds out for himself or herself without being informed by an adult or teacher. To discover is to find out something that one does not know at first all by one's own effort. Discovery is a process; a way of approaching problems rather than a product or a particular item of knowledge. Learning by discovery is learning to discover where the learner is actively involved in the learning process. Learners play freely and manipulate available materials directly sufficiently; rearrange information, integrate it with existing cognitive structure and then transform the integrated combination to generate the desired end product.

In a **pure discovery learning**, the principal content of what is to be learnt is not given but must be discovered by the learner before incorporating it into the cognitive structure (schema). The emphasis is on the kind of processes learned like problem solving skills. Discovery learning starts with some formal intuitive thinking. Learners have the intrinsic curiosity to learn and this must be channeled into discovery. The learner must be taught to participate in the process that makes possible the establishment of knowledge and the development of skills. This makes learners to think mathematically for themselves and take part in the process of attaining knowledge and skills.

Discovery Teaching, also called "Socratic method" is as old as formal education itself, since the time of Plato. It originated from a dialogue between Socrates and a slave boy. The method evolves as dialogue between teacher and learner in which a learner reaches the desired conclusion through a carefully arranged sequence of questions. Teaching by discovery consists of confronting a learner with a problem or seemingly incongruous situation and having the learner seek ways of solving or resolving it. In discovery teaching, the teacher presents learners with the necessary materials or tasks that would aid the discovery. The materials are carefully arranged by the teacher with a particular goal in mind to ensure that learners discover what is expected. Teachers must make sure that materials of different kinds, all of which embody the concept to be learnt, are presented to learners to experience (manipulate) i.e there should be perceptual variability so that learners would not associate the ideas they discover with the materials used only.

In **guided discovery**, the teacher presents the specific steps to be undertaken by the learner. The steps are carefully sequenced for the learners. Instructions are therefore given in sequences and each prerequisite step is attained before the next instruction is given. Bruner believed that if an individual is to perceive objects correctly then he or she must be guided so that his or her perception can be enhanced. Bruner argues that when learners are guided to discover, errors are eliminated and learning is made more meaningful. In guided discovery learning, patterns are observed and generalization made on them.
For example, discovering the general rule for an arithmetic progression (linear sequence) as $a_n = a + (n-1)d$, where a is the first term, d the common difference and a_n is the n^th^ or general term.

1. Teacher gives three or more linear sequences to groups of learners; e.g.
   (i) 3, 7, 11, 15,….; (ii) 2, 4, 6, 8, …; (iii) 1, 6, 11, 16, …with instructions (worksheet)
2. Learners find the common difference in each case.
3. Learners relate the first term to the rest of the terms in terms of number of differences added (looking for a pattern).
4. Learners verbalise what they observe. *The number of differences to multiply the common difference is one less than the number of terms*.

5. Teacher assists learners to come out with the mathematical or symbolic generalisation as the rule.

If we let the sequence 3, 7, 11, 15,…. be $a_1, a_2, a_3, a_4, ..., a_n$ then

$$a_1 = 3 \quad \text{and} \quad d = 4$$
$$a_2 = 3 + 4 = 7 \quad \Rightarrow \quad a_2 = a_1 + d$$
$$a_3 = 3 + 2(4) = 11 \quad \Rightarrow \quad a_3 = a_1 + 2d$$
$$a_4 = 3 + 3(4) = 15 \quad \Rightarrow \quad a_4 = a_1 + 3d$$
$$\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$
$$a_n = 3 + 2(n-1) \quad \Rightarrow \quad a_n = a_1 + (n-1)d$$ as the general term for an AP.

Learners go through similar process for the remaining two examples. Learners compare the observations made for the three examples and finally arrive at a conclusion that for any linear sequence $\Rightarrow \quad a_n = a_1 + (n-1)d$.

Strengths and Weakness of Discovery Learning

Advocates of discovery learning in mathematics classrooms state the following reasons:

1. Discovery learning is self-motivating. Learners have intrinsic satisfaction after achieving success (discovery).
2. It has positive transfer of learning effects to untaught situations. Both horizontal and vertical transfer take place.
3. It aids the development of problem solving skills.
4. Learner-centred education is enhanced. Learner participation is greater.
5. Because of deeper understanding retention is good after some time lapse.

Critics of discovery learning advance the following reasons why it must not be overused in the mathematics classrooms.

1. Beginning may be frustrating because of trial and error. It requires that the teacher be prepared for too many corrections, a lot of things initially discovered may turn out to be wrong (process of trial and error). Discovery learning has the potential to confuse learners if no initial framework is available.
2. Discovery learning is inefficient, it is too time consuming for all academic activities (for example mathematical operations), and there are not enough hours in a school year for students to 'unearth' everything on their own.
3. Discovery learning can become a vehicle to reject the idea that there are important skills and information that all children should learn.
4. If discovery learning is taken as an overriding education theory it is apt to produce an inadequate education.

Key Ideas

- **Discovery learning** is learning in which the learner finds out for himself or herself without being informed by an adult or teacher.
- Teaching by discovery consists of confronting a learner with a problem or seemingly incongruous situation and having the learner seek ways of solving or resolving it.
- **Guided discovery**, involves a teacher presenting specific steps to be undertaken by the learners. The steps are carefully sequenced for the learners. Instructions are therefore given in sequences and each prerequisite step is attained before the next instruction is given.

Reflection

- How has the session presented me with experiences and strategies to employ discovery learning and guided discovery in my mathematical lessons?

Conclusion

- Explain **four** reasons why it is becoming difficult for most mathematics teachers to initiate discovery teaching and learning in our schools.
- Describe the steps you would take the learners through to discover the formula for finding:
  a) Sum of first n terms of a linear sequence is $S_n = \frac{n}{2}(2a + (n-1)d)$ where *a* is the first term, *d* is the common difference and *n* is the number of terms.
  b) Volume of a cuboid with base area A and height *h* is $V = Ah$.
  c) Total surface area of a closed cylinder with base radius *r* and height *h* is $S = 2\pi r(r + h)$.
- Explain three reasons why you would initiate discovery learning in your mathematics class.

UNIT 6: ABSTRACTIONS, GENERALIZATION, SYMBOLS AND MODELS IN MATHEMATICS

This unit introduces participants to the abstraction and generalisations, symbols in mathematics and mathematical models.

Learning outcome(s)

By the end of the unit, the participant will be able to:
1. explain abstraction and generalization in mathematics;
2. explain symbols and contributions of symbols in mathematics;
3. explain models in mathematics and the uses of models.

SESSION 1: ABSTRACTION AND GENERALIZATION

In this session, we will focus on abstraction and generalisations in mathematics.

Learning outcomes

By the end of the session, the participant will be able to:
i. explain the processes learners go through to develop abstract thought.
ii. distinguish between abstraction and generalisation;

Now read on …

Abstraction and Generalization

Mathematics in general poses a challenge to many students mainly because it requires students to think abstractly. Students' mathematical experiences must progress through the process summed up by the mnemonic "ELPS", which stands for *Experience*, *Language, Pictures* and *Symbols.*

1. Experience with physical objects – seeing, feeling, tasting, holding, having 'fun' and learning about many of the properties. E.g. parallelogram, square, triangle, etc.
2. *Language* (spoken) that describes that experience, associate the sound of the word "square", "quadrilateral", and so on with the object.
3. *Pictures* that represent the experience. Picture is very different from the object itself but the child sees that they have a lot of things in common.
4. *Symbols* that generalize the experience. The symbol has no property at all in common with the sounds that we utter in saying the word "square".

Mathematical experience should also follow the same sequence to help the child to develop proper concept of abstraction.

Abstraction has to do with extracting what is common to a number of different situations while disregarding what is irrelevant and often regarded as *noise*. It is a process involving formulating generalized concepts of common properties. Abstraction is a mental activity in which we become aware of the similarities among our experiences. We make an abstraction when we realize properties common to a set of examples, when we see the commonality among the differences e.g. triangular shapes in different colours and of different types, isosceles, scalene, equilateral, right angled. For example, we abstract **"twoness"** from seeing many different items in pairs, such as 2 cars, 2 chairs, 2 pens, 2 sticks, 2 oranges, 2 bottles, etc. The type of objects does not matter. What we abstract is the twoness. Selecting say, 2 cars, 2 cats, 2 chairs, 2 tables, etc, all of which have 4 legs in teaching the concept of "twoness" can be misleading. Can you explain?

A **generalization** is a statement that holds over a set of objects. It occurs when we are able to predict that a relationship that holds for a particular sample will hold also for a more inclusive sample. In general, we take a simple proposition and extend it to a more elaborate theorem of which the simple proposition is a special case. It is the process of extending a class to include new situations. Further, the simple result plays a key role in the generalization proof. Generalization, therefore, is "the forming of a general notion or proposition obtained by induction. Generalizing is the ability to describe a mathematical pattern in a concise and easily understood way. It is powerful because it enables us to move from a few specific examples to a wider, more, general, statement. For example, Sum of angles in a polygon based on number of triangles the polygon has is given by $\mathrm{S} = (n-2) \times 180°$. In $(n-2) \times 180°$, $\boldsymbol{n}$ can vary depending upon the shape of the polygon. The extension of the rule an infinite number of cases is followed by a sense of wonder and the release of energy and a feeling of power. For example, for sum of interior angles of a pentagon; $(5-2) \times 180°$, for a hexagon, $(6-2) \times 180°$, etc. The use of the mathematical shorthand of letters and brackets (i.e. algebra) enables us to describe the total number of degrees in any polygon. This is the power of generalization.

Algebra involving variables is introduced in high school because children find it quite difficult to understand until age 13 or 14. Primary school equations are usually given in the form $7 + \square = 12$ so that pupils understand that they are only required to find the missing number often referred to as *unknown*. A mathematical statement that states a relation between two names for numbers, e.g. $36 + 18 = 54$ is referred to as a *number sentence*. A symbol that is not a numeral but represents a number is called a variable. In $y = 2x + 1$, x and y are **variables** because they can be many different numbers. A *solution* is a replacement for a variable in a sentence that makes the sentence true. Some solutions to $y = 2x + 1$ are $(0,1)$, $(1,3)$, $(2,5)$. The set of all possible solutions is called the *solution set*.

We notice that generalization is a way of building relationships. Mathematical formulae are generalizations. The formula $A = l \times w$ shows the relationship between Area of a rectangle and the measure of the sides, length (l) and width (w). The formula for Volume (V) of a sphere given by

$V = \frac{4}{3}\pi r^3$, shows the relationship between the volume and the radius of the sphere. Similarly, Surface area of a Cone given by $S = \pi rl + \pi r^2 = \pi r(l + r)$ shows the relationship between the area and the radius and slant height of the cone.

The definition, "*A trapezium is a four-sided figure with one pair of opposite sides parallel*" is also a generalization made from a number of observations about all available trapezia and which holds for any trapezia.
Mathematics teachers are to ensure that when teaching generalization students are encouraged to:

- see or learn the relationship by carrying out the activities
- state the relationship (by writing it)
- use the relationship by applying it to related situations
- demonstrate the validity of the relationship by verifying or proving using specific examples e.g. Show that….

Key Ideas

- Abstraction has to do with extracting what is common to a number of different situations while disregarding what is irrelevant and often regarded as *noise*.
- Abstraction is a mental activity in which we become aware of the similarities among our experiences. It involves holding of several pieces of evidence in the mind.
- A generalization is a statement that holds over a set of objects. It occurs when we are able to predict that a relationship that holds for a particular sample will hold also for a more inclusive sample.
- Generalizing is the ability to describe a mathematical pattern in a concise and easily understood way.

Reflections

- What experiences or examples have the session provided to enhance my capacity to teach abstraction and generalization in a JHS classroom?

Discussion

- With **one** illustrative example in each case, distinguish between *abstraction* and *abstracting* in mathematics.
- With **two** mathematical examples, explain what is meant by generalization in mathematics.

SESSION 2: SYMBOLS IN MATHEMATICS

In this session, we will focus on symbols and its contributions to the teaching and learning of mathematics.

Learning outcomes

By the end of the session, the participant will be able to:

i. explain symbols in mathematics;
ii. discuss the contributions of symbols to the teaching and learning of mathematics.

Now read on …

Definition of Symbols in Mathematics

A symbol is a sign that reminds us of something apart from the sign itself. It calls to attention a certain imagery which accompanies the class of things it symbolizes. A symbol is defined as a sound or something visible, mentally connected with an idea. This idea is the meaning of the symbol. A symbol represents a mathematical entity. Symbols are sort of *primitive computing machine* that will do some of the thinking for us. It is a sort of code that needs decoding.

Contributions of Symbols

1. Symbols are used for communication:

The process of making an idea conscious is closely associated with the use of symbols. A concept is a purely mental object; inaudible and invisible; illusive and inaccessible. No one can see or hear someone else's mental images. The direct way of observing the contents of our minds is by spoken words or other sounds, written words or other marks on paper. Symbols play a key part in bringing ideas into consciousness and manipulating them. The clearer the symbols the more effective they will do the job e.g. the use of ***t*** for ten in base twelve and the use of ***A*** for Area in measurement.

a) *Symbols are used to represent concepts*: Symbols are essential ingredients of mathematics that condense a hierarchy of concepts into manageable form. This tends to make mathematics abstract, e.g. $V = \pi r^2 h$.

b) By the use of symbols we are able *to evoke a concept from our memory store into consciousness* so as to make it available for reflection, e.g. 3, $4x$, π, etc.

c) The use of symbols enables us *to arrange and rearrange our mathematical ideas*. E.g. to find the simple interest, the formula $I = PTR$ comes to mind. Each symbol I, P, T, R has been chosen to evoke the appropriate concept (interest, principal, time, and rate). The way we arrange them evokes the appropriate mathematical operations by which we can obtain the answer to our problem.

When we reflect on our schemas and then make public the symbols attached, we can enable someone else to organize his own thoughts according to the same schema. Often what is meaningful to oneself may not be meaningful to the hearer. We may think we are communicating when we are not; and it is impossible to know for certain whether we are, and if so to what degree. Mathematics teachers should try to get as near as possible to the impossibility of producing the same idea in the mind of the receiver as of the communicator. Mathematics teachers should bear in mind that:

(i) a symbol and its associated concept are two different things; and
(ii) this distinction is non-trivial, being that between an object and the name of that object.
If an object is called by another name, we do not change the object itself, e.g. five, 5, v, $(101)_{\text{two}}$, etc, all name the same number in different notations.

There are three categories of **hearers (audience)** in a class:

(i) Those who do not know what we are talking about but want to (beginners). Here we need to choose the symbols with care, and use them as accurately as we can so that we communicate nothing but the truth (though not necessarily the whole truth). Concepts are built up by degrees and the first approximation is bound to be incomplete and need to be tidied up later. A brief and accurate statement may be confusing even to an intelligent learner than a somewhat lengthier, but accurate statement.

(ii) Those who know what we are talking about, as a general background, within which we are communicating some particular aspect. Here we can take much for granted, save time and concentrate on essentials. We do not have to start them from the scratch.

(iii) Those who know what we are talking about but want to fault it - to disprove it. Brilliant students in class may fall under this category. They need enrichment activities but not a repetition of what they already know.

2. **Symbols are used for explanation**

An explanation is a communication intended to enable someone to understand something, which he did not understand before. Failure to understand may be a result of:

(a) A wrong schema being used. We use explanation to activate the appropriate schema.

(b) The gap between the new idea and the existing schema may be too great, e.g. from $a^2 = a \times a$, $a^3 = a \times a \times a$ then, $a^2 \times a^3 = a^6$ and then to $a^m \times a^n = a^{m+n}$ as a generalization, has a big gap to fill.

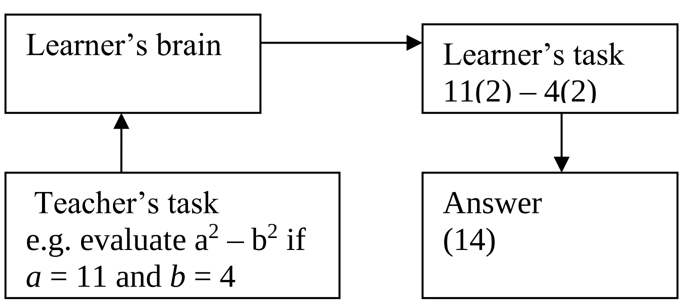
We use explanation to supply the intervening steps to bridge the gap.

(c) There may be a need for accommodation i.e. the process of modifying the existing schema. Explanation is to help to reflect on this schema; to detach it from the original examples and to modify it.

Having to explain our ideas to someone else forces us to become more consciously aware of them ourselves and sometimes to realize that we understand them less well than we thought we did. This may lead eventually to greater understanding. Students who are given the chance to practise by explaining what is learnt to others are likely to perform better in a test on that topic than those who practise on their own (and alone). Getting students to explain and discuss their ideas is one way of encouraging the development of reflective activity.

3. Symbols for recording knowledge, both for the benefit of others and of ourselves.
The ideas of the gifted mathematicians can be made available to as many of us as possible to read and understand their work when they are recorded. Knowledge builds on knowledge and each generation learns the schemas of earlier generations and among each generation. This transmission of knowledge is made possible by the use of symbols. Written symbols are used to aid our memory both in the short term and long term. Symbols recorded can be used years later to recover the ideas from our own memory stores. Ther is a permanent record for revision and checking of earlier points.

4 Symbols for reflective thinking
Symbols also function as a device which ensures that the activity of **reflective thinking** is not made impossible by the unavailability of the objects (of thought) on which they act. For example, A task is given → it enters the learner's brain →learner reflects on the task and he/she understands it in his/her own way → S/he provides an answer.



The answer here is the solution to task difference between the product of 11 and 2 and, 4 and 2, interpreting squaring as multiplying by 2. Learners' answer shows the way they understand the questions or tasks. Their tasks therefore, differ from ours. Symbols help to isolate ideas and to reflect on them before we get answers.

5 Symbols are used to make multiple classifications straightforward

Symbols enable us to bring out different properties of the concept, e.g. 9 can be written variously as 3^2, $\sqrt{81}$, (5 + 4), $18 \div 2$, etc. but each gives a slightly different property of 9.

Some of these properties may be hidden and the equality sign is used to reveal them. E.g. can you show that the expression $4x^2 - 12xy + 9y^2$ in the equation, $4x^2 - 12xy + 9y^2 = (2x - 3y)^2$ is always positive?

Other contributions of mathematical symbols include:

(a) For the formation of new concepts - introducing new concepts.

(b) Helping to show structure and to suggest ways in which individual ideas are related e.g. $f(x) = x^2$, the use of Venn diagrams to illustrate set problems.

(c) Making routine manipulations automatic.

(d) Recovering information and understanding.

(e) Creative mental activity.

Key Ideas

- A symbol is defined as a sound or something visible, mentally connected with an idea. This idea is the meaning of the symbol.
- A symbol represents a mathematical entity, e.g., ***dy/dx*** represents one meaning and ***f(x)*** represents another.
- Symbolism is a powerful mechanism for conveying mathematical ideas to others and for "doodling around" with an idea as we do mathematics.
- There are several contributions of mathematical symbols to the teaching and learning of mathematics at all levels of education.

Reflections

- How has the content of the session exposed me to the important contributions of symbols to the teaching and learning of mathematics?

Conclusions

- With ***two*** mathematical illustrations, explain what is meant by *symbols* in mathematics.
- Discuss four contributions of mathematical symbols.

SESSION 3: MATHEMATICAL MODELS

In this session, attention will be on mathematical models and its uses.

Learning outcomes

By the end of the session, the participant will be able to explain mathematical models and their uses.

Now read on …

Meaning of Mathematical Models

A mathematical representation of a practical situation is called a **mathematical model.** A model for a mathematical concept refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed. Before you can "see" in a model the concept that it represents, you must already have that concept – that relationship – in your mind, otherwise you would have no relationship to impose on the model. Models are often more meaningful to the teacher than to students because the teacher already has the concept and can see it in the model. A student without the concept sees only the physical object. The student needs to know the relationship before imposing it on the model.

Models serve as a testing ground for emerging ideas; they are seen as "*thinker toys*", "*tester toys*" and "*talker toys*". Models give students something to think about, explore with, talk about, and reason with. There are five different representations of mathematical ideas. These are: manipulative models, pictures, written symbols, oral language, and real-world situations. Translating between and within each goes a long way to help students develop new concepts.

a. The concept of "length" could not be developed without making comparisons of the length attribute of different objects. The length measure of an object is modelled as a comparison relationship of the length of the object to the length of the **unit**.

b. The concept of "rectangle" is a combination of spatial and length relationships. By drawing or modelling on dot paper, the relationship of the opposite sides being of equal length and parallel and the adjacent sides meeting at right angles can be illustrated.

c. In probability, the experiment of flipping a coin and obtaining 52% heads describes a real-world occurrence. The mathematical procedure of calculating the probability of heads as ½ is a mathematical or theoretical model. A model is good insofar as it is consistent with other real –life occurrences. A probability model deals with situations that are random in character and attempts to predict the outcomes of events with certain stated or known degree of accuracy. For example, tossing a coin, intuition tells us that it is equally likely to be head or tail, and somehow we sense that if we repeat the experiment a large number of times, the head will occur "about half the time".

d. The use of **matrices** to systematize and handle a large number of variables and real-life data is a mathematical model e.g. A company produces two models of wireless and manufactures them at four factories. The number of units produced at each factory can be summarized in matrix form as:

$$\begin{array}{r c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{r} \text{Standard model} \\ \text{Deluxe model} \end{array} & \begin{bmatrix} 16 & 25 & 15 & 8 \\ 10 & 4 & 14 & 5 \end{bmatrix} \end{array}$$

e. The use of **formulae** for solving real-life problems is a mathematical model. For example*, the Highway department plans to build a picnic area for motorist along a major highway. It is to be rectangular with an area of 5,000 square metres and it is to be fenced off on the three sides not adjacent to the highway. Express the number of metres of fencing required as a function of the length of the unfenced side.*

Naturally, we let l and w be the length and width of the picnic area. Now the function becomes $P = l + 2w$.

But required area is 5,000 implies $lw = 5{,}000$.

Solving for w we have $w = \dfrac{5000}{l}$

Substituting w into P implies $P(x) = l + 2\dfrac{5000}{l} = l + \dfrac{10{,}000}{l}$.

Uses of Mathematical Models

It is important that you have a good perspective on how manipulatives can help or fail to help students construct ideas. Models have **three** related uses in the classroom:

(i) **To help students develop new concepts or relationships**

Models help students to think and reflect on new ideas and so a variety of them must be accessible for students to select and use freely. Do not force students to use a particular model because what works for you may not work well for the student. Allow students to use their own model to go through the test-revise-test –revise process until the new concept is formed.

(ii) **To help students make connections between concepts and symbols**

Students need some guidance to make this link between concepts and symbols. Ask students to write down what they have done with models. E.g. "write an equation to tell what you just did." "How would you go about recording what you did with the models?" For example, for "twice a number less by 3" can be recorded as $2x - 3$.

(iii) **To assess students' understanding**

When students are allowed to use models in ways that make sense to them, how they use the physical models is a wonderful window into their minds. Classroom observation of your students then becomes a **student-by-student assessment**. Guide students to explain ideas with manipulative materials. Encourage students to draw pictures to help them show what they are thinking.

There is danger in teacher forcing students to "*do as I do*" in using models. Students will blindly follow your directions and it may even look as if they understand; e.g. 'invert and multiply'. This does not promote thinking, nor does it aid in the development of concepts. Students begin to use the models as answer – getting devices rather than as thinker toys.

Key Ideas

- A mathematical representation of a practical situation is called a **mathematical model.**
- A model for a mathematical concept refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed.
- Models have three related uses in the classroom. It helps students to: develop new concepts or relationships, make connections between concepts and symbols, and assess students' understanding.

Reflections

- How has the session equipped me the relevant information to teach and use mathematical models in my mathematics lessons?

Discussions

- Explain what is meant by *mathematical model* with **two** illustrative examples.
- Discuss **three** usefulness of mathematical models.

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