Module for Bachelor of Education Programme (Primary)

EBS 210SW:

ALGEBRAIC THINKING

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UNIT 1: COMPONENTS OF ALGEBRAIC THINKING

This unit discusses the definition and components of algebraic thinking. It also introduces participants to some ways of helping students to develop algebraic thinking and suggested strategies for teaching algebraic thinking.

Learning outcome(s)

By the end of the unit, the participant will be able to:
1. explain what algebraic thinking is;
2. explain useful strategies for helping students to develop algebraic thinking;
3. explain the generalisation component of algebraic thinking;
4. explain the concept of equality in algebraic thinking and distinguish between Algebraic expressions and equations;
5. distinguish between unknown and variable in mathematics; and
6. explain some strategies for teaching algebraic thinking.

SESSION 1: DEFINITION OF ALGEBRAIC THINKING

In this session, we will focus on the meaning of algebraic thinking.

Learning outcomes

By the end of this session, the participant will be able to explain what algebraic thinking is.

Now read on …

Algebra uses letters or other symbols in place of values, and then plays with them using special rules. Algebra is, in essence, the study of patterns and relationships; finding the value of x or y in an equation is only one way to apply algebraic thinking to a specific mathematical problem. We will notice that the potential for students to think algebraically resides in many of the arithmetic problems they regularly do in school. This requires only a shift in language or a slight extension of a basic arithmetic problem to open up the space of algebraic thinking for students. We shall come to realise that it is how you solve a problem that tells whether you are doing algebra or arithmetic, not the problem itself. This session discusses some definitions of algebraic thinking.

Definition of Algebraic Thinking
Algebraic thinking is *the use of any of a variety of representations that handle quantitative situations in a relational way*. Note that, the representations are to be **varied**, and the handling of the quantitative situations is to be done in a **relational** way. Steele and Johanning, (2004) define *algebraic thinking* as *the capacity to represent quantitative situations so that relations among variables become apparent* (p. 65). Algebraic thinking is the *thinking that involves representing, generalizing, and formalizing patterns and regularity in all aspects of mathematics* (Van de Walle,

2004). We can interpret this definition to mean that algebraic thinking is about *generalising arithmetic operations and operating on unknown quantities*. It involves *recognising* and *analysing patterns* and developing generalisations about these patterns. Seeing a pattern usually leads to making some generalization, and generalisations naturally come from patterns we recognize. The patterns could be extended to include patterns in all aspects of mathematics; shapes, colours, positions in sequences. This leads to extending the definition of algebraic thinking thus:

Algebraic thinking involves the construction and representation of patterns and regularities, deliberate generalization, and most important, active exploration and conjecture. (Kaput, NCTM, 1993).

Take note of the inclusion of the terms, *representing* patterns and regularities observed, and active *exploration* as important processes. These make it easy for us to make conjectures and generalizations and to verify them.

We can draw a conclusion from the definitions that algebraic thinking is *the ability to operate on an unknown quantity as if the quantity was known, in contrast to arithmetic thinking which involves operations on known quantities*. We now recognise that algebraic thinking goes beyond just learning how to work out for the variables x and y. Algebraic thinking helps students to think about mathematics at an abstract level, and provides them with a way to reason about real-life problems. It also has problem solving as a point of reference for thinking about algebra and how problem solvers model problems.

Algebraic Thinking in Algebra

In order to be more content-specific, let us define algebraic thinking as an approach to thinking about quantitative situations in general and in a relational manner. This demands a considerable understanding of the *objects of algebra*, a *disposition to think in generality*, and *engagement in high-level tasks* which provide contexts for applying and investigating mathematics and the real-world. We consider these three as the ingredients of algebraic thinking in algebra (http://keepingmathsimple.wordpress.co).

Objects of Algebra refer to the content of algebra which can be classified into three overlapping categories. Category 1 objects refer to those concerned with *representing changing and unchanging quantities and relationships*. Ideas of variables, numbers, graphs, equations, matrices, etc fall into this category. Category 2 objects refer to ideas involved in *working with unknown quantities* which cover solving equations and inequalities. Linear equations and inequalities in one or more variables, exponential, quadratic, and trigonometric equations, etc are examples of Category 2 objects. Category 3 objects are those ideas for *investigating relationships between changing quantities* which include directly and indirectly proportional relationships; relationships with constant rate and changing rate of change; relationships involving exponential growth and decay; periodic relationships, etc.

Thinking disposition draws our attention to the fact that the thinking processes that contribute to the development of algebraic thinking require purposeful representations of quantities and relationships, multiple interpretations of representations, finding structures, and generalization of patterns, operations and procedures. It is important to gain computational fluency in simplifying, transforming, and generating expressions.

High-order tasks include problem solving, mathematical investigations which are most often open-ended problem solving tasks and modeling. Teachers must therefore endeavour to challenge students with high order tasks,

Key ideas

- Algebraic thinking is the use of any of a variety of representations that handle quantitative situations in a relational way.

- Another definition of algebraic thinking is that, it is *the* "*capacity to represent quantitative situations so that relations among variables become apparent*" (Steele & Johanning, 2004. p.65)

- Algebraic thinking is defined as the thinking that involves representing, generalizing, and formalizing patterns and regularity in all aspects of mathematics (Van de Walle, 2004)

- Algebraic thinking involves the construction and representation of patterns and regularities, deliberate generalization, and most important, active exploration and conjecture (Kaput, NCTM, 1993).

- The ingredients of algebraic thinking are objects of algebra, thinking disposition and higher-order tasks.

Reflections

- How has my understanding of algebraic thinking been challenged to adequately prepare and teach mathematics lessons in the classroom?

Discussions

- Discuss the three objectives of algebra
- Discuss the definitions of algebraic thinking
- Discuss the main ingredients of algebraic thinking.

SESSION 2: ALGEBRAIC HABITS OF MIND

Dear participants, we shall now discuss a very important aspect of algebraic thinking which has to do with certain behaviours regarded as critical in developing algebraic thinking in particular and problem solving skills in general.

Learning outcome

By the end of this session, the participant will be able to explain the algebraic habits of mind useful for developing algebraic thinking

Now read on …

Mathematical Habits of Mind

Our disposition toward behaving intelligently when confronted with problems in life, for which we do not have immediate answers is often referred to as *habits of mind*. *"*A habit is any activity that is so well established that it occurs without thought on the part of the individual.*"* This is evident when an individual's way of doing a particular thing has become so automatic that they do the things without the teacher or an adult still having to ask "Can you explain why you do it that way?" or "Can you do it another way?".

Two components of algebraic thinking are the development of mathematical thinking tools and the study of fundamental algebraic ideas. *Mathematical thinking tools* are analytical habits of mind such as problem solving skills, representation skills, and reasoning skills, which help us to make sense of situations. *Fundamental algebraic ideas* represent the content domain in which mathematical thinking tools develop. These include patterns, variables, functions, generalized arithmetic, and symbolic manipulation.

Some important mathematical habits of mind that every teacher should set as targets for any mathematics lesson include:

1) Searching for Patterns: Our students should be guided to develop the habit of:

a) generating cases and generalizing patterns;
b) looking out for short-cuts that arise from patterns in calculations;
c) investigating special cases, extreme cases from patterns observed.

2) Reasoning: Teachers should encourage students in relevant activities that would help them develop the habit of:

a) explaining the positions they take;
b) providing mathematical evidence or justification for the conjectures or generalizations they make;
c) testing conjectures by generating cases both special and extreme;
d) justifying why a generalization will work for all cases or for some cases only.

3) Solving and posing problems: Let us guide students to develop the habit of:

a) always looking for alternative solutions to problems;
b) extending problems and solutions to more general case;
c) solving problems algebraically, geometrically, numerically;
d) asking clarifying and extending questions.

4) Making connections: Help students to develop the habit of:

a) linking algebra, number, geometry, statistics and probability;
b) finding or devising equivalent representations of the same concept;
c) linking mathematics concepts to real-world situation.

5) Communicating mathematically: Students should develop the habit of:

a) using appropriate notation and representation;
b) noticing faulty, incomplete or misleading use of numbers.

6) Reflecting and self-directing learning: Students should be encouraged to develop the habit of looking back to justify their solutions.

Developing Algebraic Habit of Mind

The ability to think about functions and how they work, and to think about the impact that a system's structure has on calculations are facilitated by three habits of mind often referred to as algebraic habits of mind. These are, Doing-undoing, Building rules to represent functions, and Abstracting from computation.

1) **Doing-Undoing:** This has to do with students trying to undo mathematical processes as well as do them (backtracking). That is, working backwards from the answer to the starting point. Reversibility plays a key role in effective algebraic thinking. This helps students to deeply understand the problem. Once a student can solve the equation $2x^2 - 8 = 0$ he/she should be able to answer the question "Write an equation that has solutions -2 and 2 ".

The following questions serve as a useful guide: Which process reverses the one I am using? What if I start at the end? How is this number in the sequence related to the one that came before?

2) **Building rules to represent functions:** This has to do with recognizing patterns and organizing data to represent situations in which input is related to output by well-defined functional rules. For instance, think of a number game such as *Take an input number, multiply it by 4 and then subtract 3,* is naturally a complement of doing-undoing habit of mind. The capacity to understand how a functional rule works in reverse generally makes it a more accessible and a useful process.

Teachers can use the following questions to guide: Is there a rule or relationship here? How does the rule work and how helpful is it ? How are things changing? Can I write down a mathematical rule? When I do the same thing with different numbers, what still holds true? What changes? How does the numbers in the equation relate to the problem context?

3) **Abstracting from computation:** Abstracting from computation has to do with thinking about computations independently of particular numbers that are used. Thinking algebraically involves being able to think about computations freed from the particular numbers they are tied to in arithmetic, that is, abstracting system regularities from computation. This habit of mind comes into play when students are able to realise for example that they can regroup numbers into pairs that equal 101 to make the following computation simpler:

"Compute: 1 + 2 + 3 + … + 100." (Refer to **Gauss'** approach). Do you recognise that 101 can be decomposed into 100 +1; 99 + 2; 98 + 3; and so on.

Here are some guiding questions: How is this calculating situation like/unlike that one?
How can I predict what is going to happen without doing all the calculations?
When I do the same thing with different numbers, what still holds true? What changes?
How does this expression behave like the other one?

The Role of Classroom Questions

Classroom questions posed by the teacher during instruction are very paramount in developing algebraic thinking. The following are some suggestions for making effective use of questions to help students develop algebraic habits of mind in class.

1) Teacher should consistently model algebraic thinking by making **explicit** what students might have left implicit in their thinking whenever the teacher is summarizing student responses to a mathematical activity.

2) Teacher should give **well-timed pointers** to students to help them shift or expand their thinking. That is, teacher should give hints or suggestions for extension at appropriate times. This enables students to pay attention to what is important. For example, "Once you have made a chart or table, look for an easier way, check how the numbers group and how the grouping might suggest an easier way".

3) Teacher should make it a habit to ask a **variety of relevant questions** to help students to organize their thinking. Pose questions that challenge students to analyse expressions. E.g., "Can you explain what the 5 and 3 represent in the equation $y = 3x + 5$?"

For each question that the teacher asks, there is the need to make the **intention** and the **mathematical context** clear to both the teacher and students. Make sure that the intentions of the questions asked should be balanced and the questions are asked in situations that are patently algebraic. Students' algebraic potentials are likely to go unexploited unless the teacher asks questions that are used to extend students' thinking about the problem. The teacher thus has to:

a. Reverse a routine calculating task to challenge students to undo as well as do: "Now that you can handle the factor tree, what whole numbers have three factors?"
b. Ask "what if" questions to extend beyond a single situation to a more generalized situation.
c. Exploit calculating situations in which there is a regularity, to challenge students to use calculating shortcuts based on the regularity. E.g.,
   "Without writing out all the numbers and adding them, find the total:
   $1 + 2 + 3 + 4 + \ldots + 26 + 27 + 28 + 27 + 26 + \ldots + 4 + 3 + 2 + 1$."?
d. Exploit calculating situations in which there is regularity, to challenge students to make general statements. (e.g., Think of three consecutive integers and multiply them. Does 2 divide any such product? Why? What other integers divide any such product? What is the largest integer that you can be certain divides any such product evenly? Why?)

Key ideas

- Our disposition toward behaving intelligently when confronted with problems, for which we do not have immediate answers is often referred to as *habits of mind*.
- Important mathematical habits of mind that teachers can use or set as learning targets include searching for patterns, reasoning, solving and posing problems,

making connections, communicating mathematically, and reflecting and self-directing learning.
- Mathematical thinking tools are analytical habits of mind such as problem solving skills and reasoning skills.
- Three habits of mind that are critical to developing power in algebraic are doing-undoing, building rules to represent functions and abstracting from computation.
- Teachers' classroom questions posed during instruction are very paramount in developing algebraic thinking.

Reflections

- How has the content of the session equipped you with effective classroom questioning strategies to help students develop algebraic habits of mind?

Discussions

- What is meant by mathematical habit of mind?
- Explain six mathematical habits of mind the mathematics teacher should aim to aim at achieving in lessons.
- Explain each of the following algebraic thinking habits of mind:

a) Doing-undoing
b) Building rules to represent functions
c) Abstracting from computation

- What role should the mathematics teacher play in order to make effective use of questions to help students develop algebraic habits of mind?

SESSION 3: GENERALISATION IN ALGEBRAIC THINKING

Components of algebraic thinking that provide a useful framework for recognizing whether students are thinking algebraically, and for determining whether a problem can be viewed algebraically are: (1) making **generalisations**, (2) conceptions about the equals sign (**equality**), and (3) thinking about **unknown** quantities. Teachers need to explore these three components of algebraic thinking effectively with their students in mathematics classroom. In this session, we shall discuss the generalisation component of algebraic thinking.

Learning outcome

By the end of the session, the participant will be able to to explain the concept of generalisation as a component of algebraic thinking.

Now read on …

One important goal of mathematics education is for our students to develop the skill of making generalizations and make it part of the their mental disposition or habit of mind in learning and dealing with mathematics. Learning mathematics is the best context for developing the skill of making generalizations and this is one good reason for including mathematics in the school curriculum.

What is generalization?

One definition of generalization connects it with a synonym for *abstraction*. Generalisation is the process of *finding and singling out of properties in a whole class of similar objects*. Generalisation is also defined to cover the *extension of an existing concept or a mathematical invention* as in the famous example of the invention of non-euclidean geometry. Generalization is also explained in terms of its product. For example, suppose the product of abstraction is a concept. Then the product of generalization is a statement relating the concepts, eventually referred to as *a theorem*.

Another definition is explicitly connected to the notion of patterns. The ability to discover and replicate mathematical patterns is important throughout mathematics. We often investigate numerical and geometric patterns and express them mathematically in words or symbols. We analyze the structure of the pattern and how it grows or changes, organise and analyse this information systematically, and develop generalisations about the mathematical relationships in the pattern. Our students can have meaningful experiences with generalizing about patterns, even though they may not usually express their mathematical ideas using variables and standard functions.

For example, when exploring a pattern such as 1, 3, 5, 7, 9,…, students may make the following observations:

(i) "If you add 1 to an even number, you always get an odd number"
(ii) "If you add 2 to an odd number, you always get another odd number"
(iii) "If you start at 1 and keep adding 2, you get all the odd numbers"
(iv) "If you can separate a number into two equal groups, it's an even number. If one is left over, it is an odd number."

These observations are ways of thinking about a simple pattern—the progression of positive odd integers. They also provide evidence of algebraic thinking, because each description relies on some sort of generalization that can be applied to any number.
Notice that the student is generalizing that no matter how large or small the even number, adding one (1) will create an odd number as in (i). In observation (iv), the student has identified the property that any even number can be split into equal groups, but odd numbers cannot.

These observations are examples of generalization, since they are **projecting a mathematical property onto a whole category of numbers,** "the even numbers." We need to give enough time for students to develop strategies for justifying a pattern.

The first step is **noticing that there is a pattern** in a number sequence, and then wondering if that pattern continues as the numbers get larger.
The next step is to **describe the pattern**, **followed by extending it**. Students will eventually arrive at a generalised understanding of the pattern. Then, they predict whether a specific number is part of a pattern without calculating each consecutive term.

For example, observing the pattern 1, 3, 5, 7, 9, …, students can determine that a number such as 263 is part of the pattern because it is an odd number, without writing out each odd number from 1 to 263 to be convinced of this fact. Most students will be ready to work on proving statements such as "adding 2 to an odd number produces another odd number". Students make and test conjectures and finally arrive at the generalisation that *all odd numbers are of the form* (2n +1) *where n is a whole number*.

Let us be aware of the fact that students are likely to base their generalisation on only one or two instances of a pattern, which is not enough evidence to determine whether a pattern exists. We need to explain to students that forming generalizations from only a few instances can lead to inaccurate conclusions.

Consider, the following problem.
A spider is trying to climb a wall that is 15 metres high. In each hour, it climbs up 3 metres, but falls back 2 metres. In how many hours will it reach the top of the wall? Explain your answer.

Students try to use generalization to solve this problem, and figure out that the spider climbs 2 metres total for each 2-hour period, clearly implying that it does 1 metre for each hour. Using this generalization, they come to the conclusion that it will take the spider 15 hours to reach the top of a 15-metre wall. However, while the relationship holds in general for each 2-hour period, the 14^{th} hour occurs in the middle of a 2- hour period. During this hour the spider reaches the top of the wall and climbs out, and consequently does not "slide down."

The students have made a generalization that is true in most cases, but they have neglected to notice that their current problem is an exception to the general rule of *up three, down two*. Their understanding of the relationship actually misleads them into solving the problem incorrectly.

Key Ideas

- One of the important components or frameworks of algebraic thinking is making generalisations.
- The term generalisation can be explained in several ways. For example, it can be explained in terms of its product, its connection to the notion of patterns, or can be explained to cover the extension of an existing concept or a mathematical invention.
- Generalisation is the process of *finding and singling out of properties in a whole class of similar objects*.

Reflection

- How has your exposure to the session broadened your understanding of algebraic thinking in the light of the framework, generalization?

Conclusion

- Explain two definitions of generalisation in mathematics.
- Write down two possible generalisations you can make about each of the following sequences of numbers:

a) 2, 4, 6, 8, …..
b) 1, 3, 6, 10, …

- Solve the spider problem and explain your solution. Investigate by changing the figures involved in the up and down movements.

SESSION 4: EQUALITY COMPONENT OF ALGEBRAIC THINKING

In the previous session, we learned about the generalisation component of algebraic thinking. This session deals with "equality" as another component of algebraic thinking. Our knowledge of balancing or maintaining a balance between two or more quantities comes to play in this discussion.

Learning outcomes

By the end of the session, the participant will be able to:

a) explain the concept of equality in algebraic thinking and
b) distinguish between algebraic expressions and equations.

Now read on …

Algebraic Expression and Equation

In algebra or number work, there is a distinction between what is called "expression" and "equation". Equations are usually considered as mathematics statements while expressions are seen as incomplete statements. For example, (7 + 6), (19 - 8), (23 × 9) and (15 ÷ 3) are **numerical** expressions. Algebraic expressions involve some unknowns or missing values or variables but are not complete statements. For example, (19 + ?), (35 - 2*x*), (28 ÷ *x*), and (7*y* + 15) are all **algebraic** expressions. Observe that mathematical expressions do not involve the use of the equality symbol.

Mathematical **equations** however, involve the use of the equality sign. For example, (19 - 8 = 11), (24 + 7 = 31), (23 × 9 = 207) and (15 ÷ 3 = 5) are numerical equations and they are **true** statements. There are two sides in each case, left hand side and right hand side of the equality sign. Each side contains a numerical expression. For instance, in (23 × 9 = 207), the left hand side has the expression (23 × 9) while the right hand side has 207. The expression, (19 + ?) can now be restated to become the algebraic equation (19 + ? = 27). Similarly, (35 - 2*x*) has its corresponding algebraic equation form to be (35 - 2*x* = 9).

An expression is basically an **incomplete** mathematical sentence. It is like any normal phrase in the English language. Equations on the other hand are more **complete**. They usually have a subject, a verb and a predicate. They possess relationships and are named 'equations' because they show **equality**. This equality is depicted with the use of the equal '=' sign. Mathematical statements with equality are equations. For example, if you say (*x* + 10 = 15) then this is an equation because it shows one type of relationship. But expressions do not show any form of relationship. The determining factor is the presence of the equal sign. When we encounter an equation, we are expected to **solve** it. Expressions cannot be solved because we do not know what relationship each variable or constant has to one another. Hence, expressions can only be simplified. An equation usually shows a **solution** or is bound to reveal its solution because of the equal sign.

Now write **two** corresponding algebraic equations for each of the following expressions:
a) $(5x + 11)$ b) $(28 \div x)$ c) $(7y + 15)$
Your answers should be similar to the following:
a) $5x + 11 = 16$; $5x + 11 = 1$.
b) $28 \div x = 4$; $28 \div x = 3\frac{1}{2}$
c) $7y + 15 = 22$; $7y + 15 = -6$

Let us now consider the arithmetic expression "6 + 23". This could just as well be stated in a problem form as "6 + 23 = ?" or "6 + 23 = ❑" or even "6 + 23 = *x*." These notations create a connection between arithmetic and the "missing value" image of algebra. The three statements indicate that there is a task to perform and that task has to do with looking for something (or a number) that is missing as indicated by ? or ❑, or *x*. It is that missing number which enables us to determine the truth of the statement. That is when we can establish a balance between the two sides in the statement. We notice that when 29 is used to represent the missing number, we can say that there is a balance or an equality has been established and so the use of the equality symbol (=) is correct.

Now what happens when we use say, 30, to represent the missing value in "6 + 23 = ❑"?
The two sides, 6 + 23 and ❑, will not balance, and so the use of the equality sign (=) will be incorrect. This is because the expression 6 + 23 is not equal to 30.

Students beginning algebra, for whom a sum such as (8 + 5) is a signal to compute, will typically want to evaluate it and then, for example, write 13 for the ? in the equation, $8+5=?+9$ instead of the correct answer of 4. When an equal sign is present, they treat it as a separator between the problem and the solution, taking it as a signal to write the result of performing the operations indicated to the left of the sign.
Now consider the algebraic statement "6 + 23 = ? + 17." This expression looks similar to the previous ones (e.g. 6 + 23 = ?) but there is one very important difference: *the number that replaces the* ? *is no longer* 29, *but a smaller number that when added to* 15, *produces* 29.

The issue resides in the meaning students assign to the "=" sign. In the case of the problem "6 + 23 = ?", the "=" can be thought of as "the result of the computation" or a "**do something**" to "6 + 23 = ?".

Another interpretation of the 'equal to' sign arises in an example such as, 6 + ? = 23. In this case, the notion of the 'equal to' sign as 'balancing' is important because it calls for determining the value that has to be added to 6 in order to give the result 23. However, in the example "6 + 23 = ? + 17," the equals sign must be interpreted differently. It is now a statement of **equivalence** between two quantities, in this case between "6 + 23" and "? + 17." Now the ? must be replaced by something other than 29, since "6 + 23" and "29 + 17" are not equivalent.

Understanding that the sign “=” requires that one expression be **equivalent** to the other is a basic tenet of algebra. Our students should be made to see a variety of problems with missing values in different positions, such as:
(i)5 + ? = 16 + 3 (ii) ? + 13 = 11 + 29 (iii) 17 + 28 = 40 + ? (iv) 18+ ? = 53

Key ideas

- Equality is an important component or framework of Algebraic thinking.
- Equations are usually considered as mathematics statements while expressions are seen as incomplete statements.
- Equations are complete mathematical sentence because they possess relationships. They are named “equations” because they show equality.
- Mathematical statements with **equality** are equations.
- Understanding that the sign “=” requires that one expression be **equivalent** to the other is a basic tenet of algebra**.**

Reflections

- How has my exposure to the session broadened my understanding on the distinction between “equations” and “expressions”? What specific knowledge have I acquired from this session to teach equations and expressions in a JHS classroom?

Discussions

- What is the difference between the following pairs of concepts?

a) Algebraic expression and numerical expression
b) Algebraic equation and numerical equation
c) Mathematical equation and mathematical expression

- Write a corresponding algebraic equations for each of the following expressions:

a) $(3x + 17)$ b) $(25 + 4x)$ c) $(12 - 3x)$

SESSION 5: UNKNOWN COMPONENT OF ALGEBRAIC THINKING

Recall that the equations $(6 + 23 = ?)$ and $(6 + 23 = x)$ demand that we **do something**. The ? or x are referred to as missing values to look for or to determine. This session deals with the third component of algebraic thinking referred to as **unknown** quantity or variable.

Learning outcome

By the end of the session, the participant will be able to distinguish between unknown and variable in mathematics.

Now read on …

Variable and Unknown

Variable and *unknown* are terms most frequently associated with algebra. In mathematics, an unknown is a number we do not know. An unknown in an equation is the variable to be solved. The solution of the equation is the value of the unknown. The belief is that the “unknown” will eventually become “known”. But it is possible for students to work with equations that include a variable that remains unknown. Most number tricks of the form, “*choose a number, multiply it by 3, add 6, divide by 3, subtract 2 and tell me the number – and I’ll tell you your original number*,” can be expressed algebraically without the need to use a specific number. The algebraic component is that the trick works for all numbers, not just a specific one for which we have to solve.

Here’s an example of a problem with an unknown quantity that remains unknown.
Suppose Abena has some number of pieces of erasers in her bowl. Ama has 3 more pieces of erasers than Abena has. Abena’s mother gives her 5 more pieces of erasers. Now who has more? How many more? Then Abena gives Ama one of her pieces of erasers. Now who has more? How many more?

Students can solve this problem without creating algebraic expressions that contain variables. They may draw a picture to represent the number of erasers Abena has (e.g. an oval), and then represent Ama’s erasers with an oval and three extra X’s. They could then manipulate the pictures without ever specifying what is in the oval. In this problem, finding the exact amount of erasers Abena has is not important, since the problem asks for a comparison between two quantities.

Abena	Abena XXXXX
Ama XXX	Ama XXX
In this diagram, the amount of erasers that Abena and Ama begin with is represented as ovals. The extra pieces that Ama has are represented as X’s. This diagram shows that Ama has more erasers, since she has 3 X’s, and Abena has none.	In this diagram, Abena has been given 5 more pieces of erasers, which are represented by X’s. Since she has more X’s (or individual pieces of erasers) than Ama, she must also have more total erasers, because the quantities in the ovals are the same.

However, some problems similar to the foregoing cannot be solved without figuring out the value of an unknown number of erasers. For example:
Suppose Abena has some number of pieces of erasers in her bowl. Ama has 3 more pieces of erasers than Abena has. If Abena gets more erasers so that she has twice as many as before, who has more erasers now? How many more?

Note that we do not have a single answer to this problem; it depends on how many erasers Abena had to begin with. So, if Abena had 2 erasers originally, Abena will now have 4, and Ama will have 5. On the other hand, if Abena begins with 5 erasers, then she will now have 10, while Ama has 8.

The difference between these two kinds of problems is subtle, but as students approach higher grades, they should be able to start making the distinction and solving them appropriately. These types of problems help develop algebraic thinking skills because they require students to think flexibly about quantities, and to learn how to compare related quantities. They also promote the idea that the relationship between two quantities (here, whether Abena or Ama has more erasers) can change depending on how the original amount is acted upon.

As students encounter more complex linear equations, they will be able to interpret the *"=" sign as an indication of equality, not as a sign requiring them to always compute something.* They will have already considered the kinds of patterns that they may now be asked to express in algebraic form. And they will be prepared to work flexibly with variables as unknown quantities rather than needing to figure out its value immediately. With these insights in hand, students will find that algebra is not a mystery, but a territory that already has familiar landmarks.

Now observe that **variables** are quantities with changing magnitude, hence can assume different values based on the application. They typically represent unknown values or values that can be changed to reflect given conditions. For example, the height and weight of a person do not remain constant always, and hence they are variables. In the algebraic equation, $x + y = 16$, x and y are the variables and can be changed. A variable in an equation can also be seen as a number that has not yet been determined. They are symbols that act as placeholders for numbers. They are usually visualized as letters and in certain cases can have more than one possible value. They are mostly used to indicate that a number in an equation or expression is not yet known.

Constant in Algebraic Equations

Constants and variables form an integral part of mathematics. They are defined as elements of equations and expressions that represent certain values. **Constants** are quantities with unchanging values. They are used to represent numbers with significance or a real number which has special properties in the context of the problem or the scenario it is used. In the equation x + 4 = 9, 4 and 9 are both constants (while x is a variable). Both constants and variables are represented algebraically by English or Greek letters. In general, constants are simply written as numbers, while variables are signified by letters or symbols. A constant can be an integer or fraction, or irrational number of interest, or any type of number. Some important constants have names and unique symbols that are recognizable throughout mathematics and sciences. The *pi* (symbolized as π) is a common constant in geometry, calculus, and other sciences.

Multiple variables can be used in the same equation, which usually increases the number of the possible values for the variables. Consider the equation: $x + 8 = y$. This equation has an infinite number of possible values for both x and y (1 and 9, 3 and 11, -2 and 6, etc). Equations with multiple variables are typically presented in a *system of equations*, or a set of multiple equations, to determine a minimum number of useful values.

Key ideas

- **Variable** and **unknown** are terms most frequently associated with algebra. In mathematics, an unknown is a number we do not know. An unknown in an equation is the variable to be solved.
- Constants and variables are defined as elements of equations and expressions that represent certain values.

- Both constants and variables are represented algebraically by English or Greek letters. In general, constants are simply written as numbers, while variables are signified by letters or symbols.
- Equations with multiple variables are typically presented in a *system of equations*, or a set of multiple equations, to determine a minimum number of useful values.

Reflections

- What are some of my experiences in teaching equations and expressions using constants and variables in a JHS classroom? How has the content of the session prepared me to competently teach equations with single or multiple variable(s) in the classroom?

Discussions

- Distinguish between unknown and variable in mathematics, illustrating with suitable examples.
- Distinguish between constant and variable in mathematics, illustrating with suitable examples.
- Write down four examples of an equation that has multiple solutions and provide possible solutions.

SESSION 6: ALGEBRAIC THINKING TEACHING STRATEGIES

This session focuses on some teaching strategies employed in teaching algebraic thinking.

Learning outcome

By the end of the session, the participant will be able to explain some strategies for teaching algebraic thinking.

Now read on …

In the transition from arithmetic to algebra, students need to make many adjustments. At present, many basic schools do not seem to focus on the representation of relations (Kilpatrick, Swafford, & Findell, 2001). In solving a problem such as *When 3 is added to 5 times a certain number, the sum is 38; find the number,* students emerging from arithmetic will subtract 3 from 38 and then divide by 5; that is, undoing the operations stated in the problem text in reverse order. In algebra classes however, students will be taught first to represent the relationships in the situation by using the stated operations; $5x+3=38$.

Observe that students operating in an arithmetic frame of reference tend not to see the relational aspects of operations; their focus is on calculating. Such students need considerable adjustment in order to develop an algebraic way of thinking. This adjustment includes the following:

1. A focus on relations and not merely on the calculation of a numerical answer.
2. A focus on operations as well as their inverses, and on the related idea of doing-undoing;
3. A focus on both representing and solving a problem rather than on merely solving it;
4. A focus on both numbers and letters, rather than on numbers alone. This includes:
    i. Working with letters that may at times be unknowns or variables;
    ii. Accepting unclosed literal expressions as responses;
    iii. Comparing expressions for equivalence based on properties than on numerical evaluation;
5. A refocusing of the meaning of the equal sign.

Teaching Algebraic Thinking without the *x*'s

We can guide our students to learn about algebra even when there are no *x*'s yet. We just need to ensure that the students deal with concepts that still make sense to them. Let us discuss five suggested ways for teaching algebraic thinking as students learn about numbers and number operations.

1) **Varying the "orientations" of the way you write number sentences**

The num ber fact 18 + 9 = 27 can also be written as 27 = 18 + 9. Notice that the first expression is about *doing mathematics*, while the second has to do with students *thinking about the mathematics.* Both expressions are different representations of the number 27. The thinking involved in the second one is identified as *algebraic*.

2) **Being mindful of the meaning of equal to sign**

Suppose we want to ask students to find the sum of 23 and 8. We can simply state it as 23 + 8 = ? But more appropriately we can write it as *What number is the same as (or equal to)* 23 + 8*?* If we want to promote algebraic thinking, we can still better state it as *What number phrases are the same as (or equal to)* 23 + 8? This is most likely to help minimize any misconception of the meaning of equal sign.

3) **Encouraging learners to generalize**

Recall that a task such as $12 + x = 15 + y$ has multiple answers. We need to encourage students to make a statement about the relationships between the numbers that satisfy the equation. Algebra is about relationships and making generalizations. Let us give a lot of opportunities to students to explain their answers.

4) **Encouraging learners to always find other ways of solving a problem**

Help students to come to realise that algebra is about relation first, and calculation second. Students may initially solve the problem ? + 8 = 7 + 19 by adding 8 and 19 then taking away 7 to find the value of ?. Now try to encourage them to find other ways of doing this. They may recognise that 7 is one less than 8 so to keep the balance, the unknown number should be *1 less than 19 which is 18.* This solution illustrates algebraic thinking.

5) **Developing the habit of investigating number representations and number relationships**

We have learnt that algebra is about generalizing arithmetical processes. Investigation activities help in making generalisation. Then a very good mathematics investigation project is to challenge students to check if the relationship for ? + 8 = 7 + 19 works for operations other than "addition".

Some Algebraic Thinking Tasks

Let us now discuss a few activities on algebraic thinking.

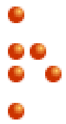
Activity 1: Generalising that $1+2+3+4+5+...+n=\frac{n(n+1)}{2}$

Showing that the ***n*-th** Triangular number, $1+2+3+4+5+...+n=\frac{n(n+1)}{2}$.

The first approach is a **visual** one involving only the formula for the area of a rectangle. This is followed by a proof using algebra.

(i) A visual proof that $1+2+3+4+5+...+n=\frac{n(n+1)}{2}$

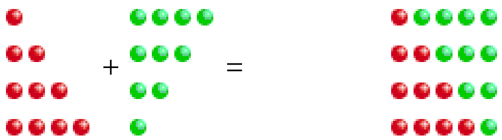
We can visualize the sum $1+2+3+4+5+...+n$ as a **triangle of dots**. Numbers which have such a pattern of dots are called **Triangle (or triangular) numbers.** Let us use the notation T(n), for the sum of the integers from 1 to n.

N	1	2	3	4	5	6
T(n) as a sum	1	1+2	1+2+3	1+2+3+4	1..5	1..6
T(n) as a triangle					…	
T(n)	**1**	**3**	**6**	**10**	**15**	**21**

For the proof, we will *count the number of dots in T(n)* but, instead of *summing the numbers 1, 2, 3, etc up to n* we will find the total using only *one multiplication and one division!*

To do this, we will fit **two copies of a triangle of dots together**, one red and an upside-down copy in green.

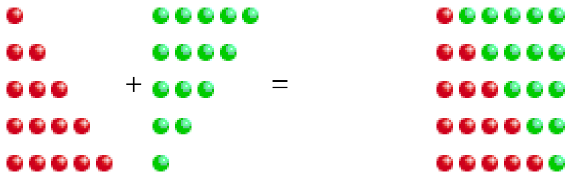
E.g. T(4)=1+2+3+4



Notice that

- we get a **rectangle** which has the same number of rows (4) but has one extra column (5)
- so the rectangle is **4 by 5**
- it therefore contains 4 x 5 = 20 balls
- but we took **two** copies of $T_{(4)}$ to get this
- so we must have 20 ÷ 2 = 10 balls in $T_{(4)}$, which we can easily check.

This visual proof applies to any size of triangle number.
Check for $T_{(5)}$:



So $T_{(5)}$ is half of a rectangle of dots 5 tall and 6 wide, i.e. half of 30 dots, so T(5)=15.

(ii) We might write out the above proof using algebra:

$$T_n = 1 + 2 + 3 + 4 + 5 + ... + n$$

$$T_n + T_n = 1 + 2 + 3 + 4 + 5 + ... + n + [n + (n-1) + (n-2) + ... + 2 + 1]$$

(Adding in reverse order for the second T(n))

$2T_{(n)} = (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (n+1)$

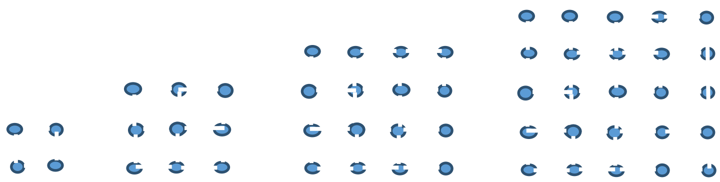
$2T_{(n)} = n(n+1)$ ……… there are 'n' terms of $(n+1)$

$T_{(n)} = \dfrac{n(n+1)}{2}$ ……………. Dividing through by 2

Hence $T_n = \dfrac{1}{2}n(n+1)$ as the generalization for triangular numbers

Activity 2: Toothpick squares (Growing squares made from toothpicks)

1) Study the pattern and draw a picture of the next likely shape in the pattern.



2) How many small squares make up the new square?
3) How many small squares would make up a large square which has 10 toothpicks on each side? Show your work.
4) Write a rule which will allow you to find the number of small squares in any large square.
5) Find a rule which will let you find the number of toothpicks in any large square. Show your work.

Key ideas

- Students operating in an arithmetic frame of reference need considerable adjustment in order to develop an algebraic way of thinking.
- Five suggested ways for teaching algebraic thinking are (1) varying the "orientations" of the way you write numbers (2) being mindful of the meaning of the equal to sign (3) encouraging learners to generalize (4) encouraging learners to always find other ways of solving a problem and (5) developing the habit of investigating number representations and number relationships.

Reflections

- How has the session equipped you with strategies to help students to develop algebraic way of thinking?

Discussions

- Explain five adjustments required for helping students operating in arithmetic frame to develop algebraic way of thinking.
- Explain five ways you can encourage students to learn algebra even without variables.

UNIT 2: PROPERTIES OF INTEGERS AND ALGEBRA OF SETS

This unit introduces participants to the study on properties of integers and algebra of sets. Focus is on properties of integers, subsets, power sets and complements of sets, properties of operations on sets and some application of sets.

Learning outcomes
By the end of the unit, the participant will be able to:

1. explain at least three properties of integers;
2. explain subsets, power sets and complements of sets;
3. explain the properties of operations on sets; and
4. solve relevant problems on applications of sets.

SESSION 1: PROPERTIES OF INTEGERS

In this session, we will focus on properties of integers. There are a number of properties of integers which regulate or determine its operations. These principles or properties help us to solve many equations. The properties will help to simplify and answer a series of operations on integers quickly.

Learning outcome

By the end of this session, the participant will be able to explain the properties of integers and solve related mathematical problems under properties of integers.

Now read on …

Integers include the set of positive numbers, zero and negative numbers which is usually represented with the letter Z.

Z = {…-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7,…}.

In the field of mathematics, mathematical equations have their own scheming principles which help us to solve such equations. The properties of integers are the rudimentary principles of the mathematical structure. There are five properties: *Closure* property, *Commutative* property, *Associative* property, *Distributive* property and *Identity* property.

Closure Property
The *closure property under addition and subtraction* states that the sum or difference of any two integers will always be an integer i.e.

$if\ x\ and\ y\ are\ any\ two\ integers,\ x+y\ and\ x-y\ will\ also\ be\ an\ integer.$

Let's look at this example that follows.

a) $4 - 5 = 4 + (-5) = -1$ and b) $(-6) + 9 = 3$

From the examples above, you will notice that the resulting difference (-1) or sum (3) are also integers.

Closure property under multiplication states that the product of any two integers will be an integer i.e. if x and y are any two integers, then xy will also be an integer.
For example, 8×5 = 40 and (–6) × (4) = −24, with results (40) and (−24), are also integers.

Division of integers doesn't follow the closure property, i.e. the quotient of any two integers x and y, may or may not be an integer. For example, (−4) ÷ (−7) = $\frac{4}{7}$, and $\frac{4}{7}$ is **not** an integer.

Commutative Property
Commutative property of addition and multiplication states that the order in which we either add or multiply given integers doesn't matter; the sum or product will be same. Whether it is addition or multiplication, swapping of terms will not change the sum or product. Suppose x and y are any two integers, then

$$x + y = y + x; \quad \text{and} \quad x \times y = y \times x.$$

For example, $6 + (-8) = -2 = (-8) + 6,$ and

$$12 \times (-5) = -60 = (-5) \times 12$$

But, subtraction and division are not commutative for integers and whole numbers. That is, $(x - y \neq y - x)$ and $(x \div y \neq y \div x)$.
For example, $4 - (-6) = 10$; and $(-6) - 4 = -10$.
Thus, $4 - (-6) \neq (-6) - 4$
For example, $10 \div 2 = 5$; and $2 \div 10 = \frac{1}{5}$.
Thus, $10 \div 2 \neq 2 \div 10$

Associative Property
We now consider the associative property of addition and multiplication which states that the way of grouping of numbers does not matter; the result will be the same. One can group numbers in any way but the answer will remain the same. Parenthesis can be done irrespective of the order of terms.
Let x, y and z be any three integers, then

$$x + (y + z) = (x + y) + z$$
$$\Rightarrow x \times (y \times z) = (x \times y) \times z$$

Example: $1 + (2 - 3) = 0 = (1 + 2) + (-3)$;
$1 \times (2 \times (-3)) = -6 = (1 \times 2) \times (-3)$

Subtraction of integers is **not** associative in nature i.e.

$$x - (y - z) \neq (x - y) - z.$$

Example: 1− (2 − (−3)) = −4; and (1 – 2) – (−3) = 2.
Thus, 1 – (2 – (−3)) ≠ (1 − 2) − (−3).

Distributive Property
Distributive property explains the distributing ability of an operation over another mathematical operation within a bracket. It can be either distributive property of multiplication over addition or distributive property of multiplication over subtraction. Symbolically we have, for any integers x, y and z:

(i) $x \times (y + z) = x \times y + x \times z$
(ii) $x \times (y - z) = x \times y - x \times z$

Example: Compute $-7\ (3 + 2)$
First approach, is to first add the terms in the bracket before multiplying by -7.

$$-7\ (3 + 2) = -7(5) = -35\ .$$

The second approach is to multiply each term in the bracket first and then add.

$$-7(3 + 2) = (-7 \times 3) + (-7 \times 2)$$
$$= (-21) + (-14) = (-35)$$

You will notice that in both approaches, we obtain the same result.

Identity Property

Among the various properties of integers, *additive identity property* states that when any integer is added to zero it will give the same number. Zero is called the **additive identity**. For any integer x, $x + 0 = x = 0 + x$

Multiplicative identity property for integers says that whenever a number is multiplied by the number one (1) it will give the integer itself as the product. Therefore, the integer 1 is called the multiplicative identity for a number. That is, for any integer x, $x \times 1 = x = 1 \times x$

If any integer is multiplied by 0, the product will be zero.
That is, $x \times 0 = 0 \times x = 0$
If any integer is multiplied by -1, the product will be opposite of the number.
Thus; $x \times (-1) = -x = (-1) \times x$

Key ideas

- Five basic properties of integers include the Closure property, Commutative property, Associative property, Distributive property and Identity property.

Reflections

- How has the content of the session broadened my understanding of the basic properties of integers to effectively teach these concepts in the classroom?

Discussions

- What is the difference between *Commutative* and *Associative* properties of Integers?
- With an illustrative example, explain *identity property* of integers.
- With two illustrative examples in each case, explain the following:
  - Distributive property of integers; and
  - Identity property of integers.

SESSION 2: SUBSETS, POWER SETS AND COMPLEMENTS OF SETS

In this session, we will discuss subsets, power sets and complements of sets.

Learning outcomes

By the end of the session, the participant will be able to guide students to explain the underlying concepts of:

a) subsets and power sets and illustrate with specific examples;
b) complement of sets and demonstrate de Morgan's laws on complement of sets.

Now read on …

Subsets

If every member of set A is also a member of set B, then A is said to be a subset of B, written A⊆B (also pronounced A is contained in B). Equivalently, we can write B⊇A, read as B is a superset of A, B includes A, or B contains A. The relationship between sets established by ⊆ is called inclusion or containment. If A is a subset of B, but not equal to, B, then A is called a proper subset of B, written $A \subset B$ (A is a proper subset of B) or $B \supset A$ (B is a proper superset of A). For example, the set of all men is a proper subset of the set of all people. Also,$\{1, 3\} \subset \{1, 2, 3, 4\}$, and $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$. The empty set is a subset of every set and every set is a subset of itself: that is, $\emptyset \subset A$ and $A \subseteq A$. Two seemingly different sets are equal; i.e.,

A = B, if and only if A⊆B and B⊆A.

Power Set

In set theory, the *power set of a* set K is defined as the set of all subsets of the set K including the set itself and the null set. It is denoted by P(K). If the given set has n elements, then its *Power Set* will contain 2^n elements. The expression 2^n represents the cardinality of power set.

For example, let Set K = {a, b, c, d}. Then number of elements is 4.
Now, the subsets of the set K are:
{ },{a},{b},{ c },{ d },{ a, b },{ a, c },{ a, d },{ b, c },{ b, d },{ c, d },{ a, b, c },
{ a, b, d }, {a, c, d },{b, c, d }, { a, b, c, d }

The power set **P(K) = {** { },{a},{b},{ c },{ d },{ a, b },{ a, c },{ a, d },{ b, c },
{ b, d },{ c, d }, { a, b, c },{ a, b, d }, {a, c, d },{b, c, d }, { a, b, c, d }**}**
Therefore, the Power Set has $2^4 = 16$ elements.
The number of elements of a power set is written as $|K|$. Thus if K has n elements then it can be written as $|P(K)| = 2^n$.

Complements of sets

If A is any set, with some universal set U defined, the complement of A, normally written as A^c or A', is defined as "all those elements that are not contained in A but are contained in U". The complement of set A with respect to the universal set are those elements in the Universal set that are not in set A. In other words, any member found in the universal set but not found in set A is described as the complement of set A. That is, $A^1 = \{x \in U | x \notin A\}$.

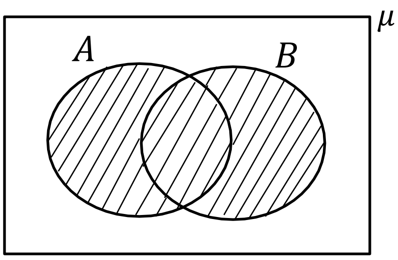
Let A and B be subsets of some universal set U. The **set difference** of A and B, or relative complement of B with respect to A, written **A−B** and read "A minus B" or "the complement of B with respect to A," is the set of all elements in A that are not in B. $A - B$ or $A \backslash B$ or A difference $B = \{x \in U | x \in A \text{ and } x \notin B\}$.
For example, let the universal set U={0, 1, 2, 3, ..., 12, 13, 14}, $A = \{0, 1, 2, 3, 9\}$ and $B = \{2, 3, 4, 5, 6\}$. Then,

a) $A-B=A\backslash B$ = {0, 1, 9}
b) A^1 = {4, 5, 6, 7, 8, 10, 11, 12, 13, 14}
c) B^1 = {0, 1, 7, …, 12, 13, 14}

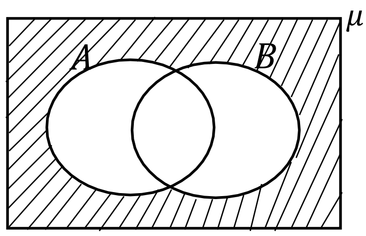
De Morgan's Laws of Complement of Sets
Augustus De Morgan (27 June 1806 to 18 March 1871) was a British Mathematician and logician who formulated the laws below in set theory. He also introduced the term "Mathematical Induction"

1. The complement of the union of sets is the intersection of the complements of the sets. Thus, $(A \cup B)' = A' \cap B'$. This law applies to any number of sets.
2. The complement of the intersection of sets is the union of the complements of the sets. Thus, $(A \cap B)' = A' \cup B'$. Again, this law could apply to any number of sets.

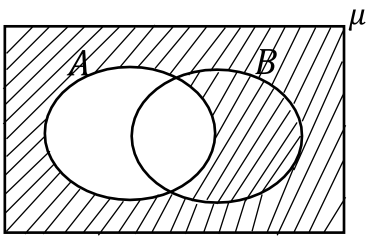
Demonstrating the proof of $(A \cup B)' = A' \cap B'$
This method demonstrates the proof using Venn diagram illustrations. To prove that $(A \cup B)' = A' \cap B'$, we will shade the two regions in a Venn diagram and compare.



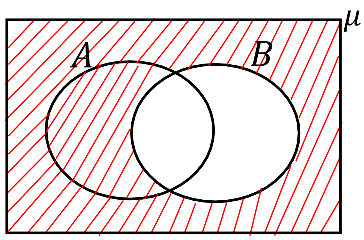
$A \cup B$- elements in A or B



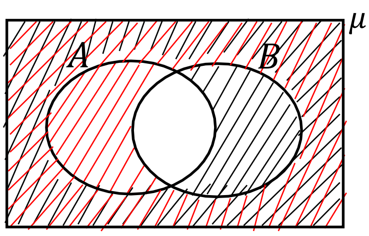
$(A \cup B)' - elements\ not\ in\ A\ or\ B$

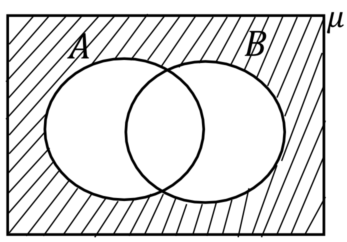


$A' - elements\ not\ in\ A$



$B' - elements\ not\ in\ B$





$A' \cap B' - Region\ with\ double\ shade$ $A' \cap B' - properly\ shaded\ out$

Comparing the regions $(A\cup B)'and\ A'\cap B'$, we can conclude they are equal.

Key ideas

- If every member of set A is also a member of set B, then A is said to be a subset of B, written A⊆B (also pronounced A is contained in B). Equivalently, we can write B⊇A, read as B is a superset of A, B includes A, or B contains A.
- In set theory, the *power set* or *power set of a* Set K is defined as the set of all subsets of the Set K including the set itself and the null or empty set. It is denoted by P(K).
- The complement of a set: If A is any set, with some universal set U defined, the complement of A, normally written as A', is defined as "all those elements that are not contained in A but are contained in U".

Reflections

- What are some experiences of teaching Subsets, Power Sets, Complement of Sets and set difference in the classroom? How has the session exposed me to definitions, explanations and examples of sets to teach in a JHS classroom?

Discussions

- Find the power set of Z = {1, 2, 5,7} and total number of elements.
- Prove De Morgan's second law using Venn diagram.
- Given A={1, 3, 5, 7, 9, 11}and B={2, 3, 5, 7, 11}, evaluate:
  a. $(A'\cap B')'$
  b. $(A'\cup B')'$
- Given that $P=\{x{:}x\ is\ even\}$ and $Q=\{x{:}\ x\ is\ a\ whole\ number\}are\ subsets\ of\ \mu=\{x{:}-1\leq x<6,\ x\in\mathbb{Z}\}$, find:

a) (i)$(Q')'$ (ii) $(P\cup Q)'$ (iii) $(P\cap Q)'$ (iv) $P'\cap Q'$ (v) $P'\cup Q'$

b) What conclusion can you draw from your results in b and d?

SESSION 3: PROPERTIES OF OPERATION ON SETS

In this session, we shall discuss properties of operations on sets.

Learning outcome

By the end of the session, participant will be able to explain the commutative, associative and distributive properties of sets with illustrative examples in each case.

Now read on…

Commutative Property

Generally, in mathematics, if an operation is commutative then it means the order of the operands (the objects being operated on) is not of great concern because the end

result whichever way will be same. The commutative property of sets holds for both union and intersection of sets. In this case, our operands are the sets and the operators are either the union or intersection.
For any two given sets A and B belonging to a universal set μ, $A \cup B$ means combining elements of sets A and B without repeating elements common to A and B. $B \cup A$ on the other hand, means combining elements of sets B and A without repeating elements common to B and A.
$A \cap B$ means finding elements common to both A and B while $B \cap A$ means finding elements common to both B and A. Thus,
a) $A \cup B = B \cup A$ ($union\ of\ sets\ is\ commutative$)
b) $A \cap B = B \cap A$ ($intersection\ of\ sets\ is\ commutative$)

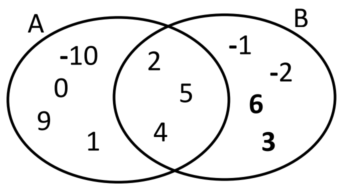
Associative Property
Associativity has to do with three operands but a single type of operator. For instance, to perform the operation, $A \cup (B \cup C)$, the property states that, it does not matter which two you operate first, the end result in any case should be same. If associativity holds for an operation, then it gives you so many choices of convenience.
For operations on sets, we can also make the following declarations:
$A \cup (B \cup C) = (A \cup B) \cup C = (A \cup C) \cup B$; (union of sets is associative)
$A \cap (B \cap C) = (A \cap B) \cap C = (A \cap C) \cap B$; (intersection of sets is associative)

Distributive Property
The distributive property under set theory makes use of three operands and two different types of operators. It aligns with the expansion property in algebra where we could make pronouncements such as $a \times (b + c) = a \times b + a \times c$. Here in set theory, we can make use of two forms of the distributive property:
a) Intersection of sets is distributive over union of sets
In symbols, we have, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
b) Union of sets is distributive over intersection of sets
In symbols, we have, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

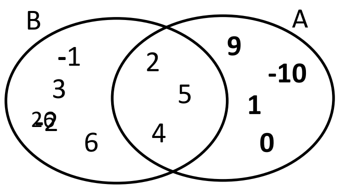
Activity 1:
Given sets $A = \{-10, 0, 1, 9, 2, 4, 5\}$ and $B = \{-1, -2, 5, 6, 2, 3, 4\}$, verify the following including the use of Venn diagram:
a. Set union is commutative.
b. Set intersection is commutative.

Solution:
(i) Let us verify that union is commutative.
$A\ U\ B = \{-10, 0, 1, 9, 2, 4, 5\}\ u\ \{-1, -2, 5, 6, 2, 3, 4\}$
$A\ U\ B = \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\}$ --------- (1)
$B\ U\ A = \{-1, -2, 5, 6, 2, 3, 4\}\ u\ \{-10, 0, 1, 9, 2, 4, 5\}$
$B\ U\ A = \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\}$ --------- (2)

From (1) and (2), we have $A\ U\ B = B\ U\ A$
By Venn diagram, we have



$A\ U\ B$



$BU\ A$

From the above Venn diagrams, it is clear that $A\ U\ B\ =\ B\ U\ A$
Hence, it is verified that set union is commutative.

(ii) Let us verify that intersection is commutative.

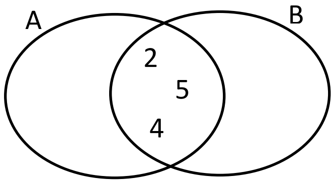
$$A\ \cap\ B\ =\ \{-10, 0, 1, 9, 2, 4, 5\}\ \cap\ \{-1, -2, 5, 6, 2, 3, 4\}$$

$A\ \cap B\ =\ \{2, 4, 5\}$ ---------(1)

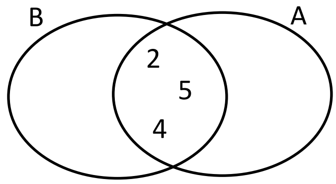
$$B\ \cap\ A\ =\ \{-1, -2, 5, 6, 2, 3, 4\ \}\ \cap\ \{-10, 0, 1, 9, 2, 4, 5\}$$

$B \cap A\ =\ \{2, 4, 5\}$ ---------(2)

From (1) and (2), we have $A\ \cap\ B\ =\ B\ \cap\ A$
By Venn diagram, we have



$A\ U\ B$



$BU\ A$

From the above two Venn diagrams, it is clear that A ∩ B = B ∩ A
Hence, it is verified that set intersection is commutative.

Activity 2
Given sets, A = {1, 2, 3, 4, 5}, B = {3, 4, 5, 6} and C = {5, 6, 7, 8}, verify that $A\ U(B\ U\ C\ )\ = (A\ U\ B)\ U\ C$.

Solution
$B\ U\ C = \{3, 4, 5, 6\} \cup\ \{5, 6, 7, 8\} = \{3, 4, 5, 6, 7, 8\}$
$A\ U\ (B\ U\ C) = \{1, 2, 3, 4, 5\}\ \cup\ \{3, 4, 5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ ---------(1)
Now, $A\ U\ B\ =\ \{1, 2, 3, 4, 5\}\ \cup\ \{3, 4, 5, 6\} =\ \{1, 2, 3, 4, 5, 6\}$
$(A\ U\ B)\ \cup\ C = \{1, 2, 3, 4, 5, 6\} \cup \{5, 6, 7, 8\}\ = \{1, 2, 3, 4, 5, 6, 7, 8\}$ ---------(2)

From (1) and (2), we conclude that $A\ U\ (B\ U\ C) = (A\ U\ B)\ U\ C$
You can also verify using Venn diagram.

Activity 3
Given sets A = {a, b, c, d}, B = {a, c, e} and C = {a, e}, verify that $A\ \cap (B\ \cap\ C) = (A\ \cap\ B)\ \cap C$.

Solution
$B\ \cap\ C\ =\ \{a, c, e\}\ \cup\ \{a, e\} = \{a, e\}$
$A\ \cap\ (B\ \cap\ C) = \{a, b, c, d\}\ \cap\ \{a, e\}\ = \{a\}$ ---------(1)
Now, $A\ \cap\ B = \{a, b, c, d\}\ \cup\ \{a, c, e\} = \{a, c\}$
Then $(A\ \cap B)\ \cap\ C = \{a\}$ ---------(2)
From (1) and (2), we conclude that $A\ \cap\ (B\ \cap\ C) = (A\ \cap B)\ \cap\ C$

You can also verify using Venn diagram.

Activity 4

Given sets A = {0, 1, 2, 3, 4}, B = {1, -2, 3, 4, 5, 6} and C = {2, 4, 6, 7} verify that $A \, U \, (B \cap C) = (A \, U \, B) \cap (A \, U \, C)$.

Solution

$B \cap C = \{1, -2, 3, 4, 5, 6\} \cap \{2, 4, 6, 7\} = \{4, 6\}$

$A \, U \, (B \cap C) = \{0, 1, 2, 3, 4\} \, U \, \{4, 6\} = \{0, 1, 2, 3, 4, 6\} ----- (1)$

Now, $A \, U \, B = \{0, 1, 2, 3, 4\} \, U \, \{1, -2, 3, 4, 5, 6\} = \{-2, 0, 1, 2, 3, 4, 5, 6\}$

And $A \, U \, C = \{0, 1, 2, 3, 4\} \, U \, \{2, 4, 6, 7\} = \{0, 1, 2, 3, 4, 6, 7\}$

$(A \, U \, B) \cap (A \, U \, C) = \{-2, 0, 1, 2, 3, 4, 5, 6\} \cap \{0, 1, 2, 3, 4, 6, 7\}$

$(A \, UB) \cap (A \, U \, C) = \{0, 1, 2, 3, 4, 6\} ---------- (2)$

From (1) and (2), we conclude that $A \, U \, (B \cap C) = (A \, U \, B) \cap (A \, U \, C)$

You can also verify using Venn diagram.

Key ideas

- The union of sets is commutative: $A \cup B = B \cup A$
- The intersection of sets is commutative: $A \cap B = B \cap A$
- The union of sets is associative: A∪ $(B \cup C) = (A \cup B) \cup C = (A \cup C) \cup B$
- The intersection of sets is associative: A∩ $(B \cap C) = (A \cap B) \cap C = (A \cap C)$
- Intersection of sets is distributive over union of sets: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Union of sets is distributive over intersection of sets: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Reflections

- How has my exposure to the content of this session exposed me with the relevant information to effectively teach properties of operations in a JHS classroom?

Discussions

1. Let μ ={1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A={1, 3, 5, 7, 9}, B={2, 4, 6, 8, 10}, and C ={1, 2, 4, 5, 8, 9}. List the elements of each of the following sets.
   a. (A∩B)∪C (b) $(A \cup B \cup C)^c$ (c) $(A \cap B \cap C)^c$
2. Let μ = {1, 2, 3, ..., 12} and K = {2, 3, 5, 7, 11}. Verify that:
   (a) K ∪ K = K (b) ϕ ∪ K= K (c) K ∪ $\mu = \mu$ (d) $\phi \cup \phi = \phi$
   (e) $K \cup K^1 = \mu$
3. Given μ = {1, 2, 3, 4,…,10}, A = {1, 4, 9}, B = {1, 3, 5, 7, 9} and C = {1, 3, 6, 10}, show that:
   a. $(A \cup B) \cup C = A \cup (B \cup C)$ b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   c. $(A \cap B) \cap C = A \cap (B \cap C)$ d. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   e. $(A \cup B) \cap (B \cup C) = (C \cup B) \cap (B \cup A)$
3. Think of a universal set and any three subsets of this universal set and demonstrate the following properties of set operations using your sets.
   a. Commutative property
   b. Associative property
   c. Distributive property

SESSION 4: APPLICATION OF SETS

In this session, we will discuss problems involving three sets, especially using Venn diagrams.

Learning outcomes

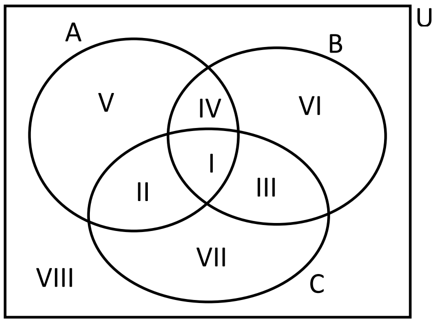
By the end of the session, the participant will be able to:

1. name the regions in Venn diagram representing three intersecting sets;
2. solve simple three-set problems.

Now read on…

Three-set Problems

We begin our discussion by learning how to describe the various regions of a Venn diagram representing three intersecting sets A, B and C.

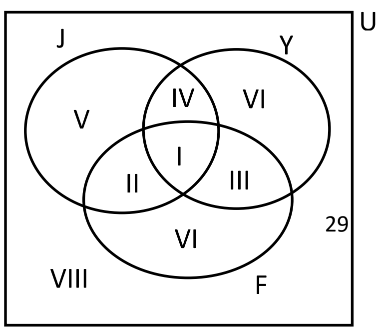


From the Venn diagram, Region I represents elements that can be found in sets A, B and C, which represents the intersection of A, B and C. Now elements in Region II are inside A and C, but outside B. This can be written as $A \cap B^1 \cap C$, the intersection of A, B^1 and C. Similarly, Region III is inside B and C, but outside A, i.e. $A^1 \cap B \cap C$. Also, Region IV is inside A and B, but outside C, i.e. $A \cap B \cap C^1$.

Now Region V is clearly inside A, but outside B and C, so we write $A \cap B^1 \cap C^1$. Similarly, Region VI is inside B but outside A and C, i.e. $A^1 \cap B \cap C^1$. Also, Region VII is inside C, but outside A and B, i.e. $A^1 \cap B^1 \cap C$. Now Region VIII is inside the universal, U, but outside Regions A, B, and C. This can be expressed as $A^1 \cap B^1 \cap C^1$.

Let's consider a real life example. In the figure below;

$$U = \{names\ of\ the\ months\ of\ the\ year\}$$
$$F = \{those\ months\ with\ more\ than\ five\ letters\}$$
$$J = \{those\ months\ which\ begins\ with\ J\}$$
$$Y = \{those\ months\ which\ end\ with\ y\}$$



1. Describe the elements of each of the regions i to viii.
2. List the elements of each of these regions.
3. Express each region in terms of F, J and Y.

I = {months with more than five letters and begins with J and end with y}
II = {months with more than five letters and begin with J, but do not end with y}
III = {months with more than five letters and end with y but do not begin with J}
IV = {months begin with J and end with y, but do not have more than five letters}
V = {months begin with J, but do not end with y and not with more than five letters}
VI = {months end with y, but do not begin with J nor have more than five letters}
VII = {months with more than five letters, but do not begin with J nor end with y}
VIII = {months which do not begin with J, or end with y or with more than five letters}

Now, the elements of each of the regions.

I = {January}
II = {}
III = {February}
IV = {July}
V = {June}
VI = {August, September, October, November, December}
VIII = {March, April}.

Now if we express each of the region in terms of F, J and Y, we shall obtain the following:

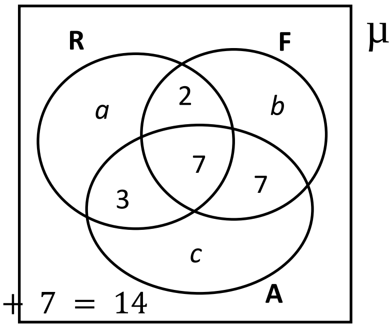
I = $F \cap J \cap Y$
II = $F \cap J \cap Y^1$
III = $F \cap J^1 \cap Y$
IV = $F^1 \cap J \cap Y$
V = $F^1 \cap J \cap Y^1$
VI = $F^1 \cap J^1 \cap Y$
VII = $F^1 \cap J^1 \cap Y^1$

Let us work through the following problem.
A number of clients who usually visit a restaurant by name Dzidzibi Restaurant were asked whether they liked rice, fufu or ampesi. Fourteen clients said they liked rice, 18 liked fufu and 23 liked kenkey. Only 7 said they liked three foods. Seven clients liked rice and fufu (this includes those that like all three foods). Ten clients liked rice and ampesi, and 14 clients liked ampesi and fufu. How many of the clients liked:

a. Rice only.
b. Fufu only.
c. Ampesi only; and
d. Rice or Fufu or Ampesi.

Let R represent those who liked Rice, F represent those who liked Fufu; and A represent those who liked Ampesi.
Then, $n(R) = 14, n(F) = 18,\ n(A) = 23,\ n(A \cap F \cap A) = 7$

$n(R \cap F \cap A^1) = 9$, $n(R \cap F1 \cap A) = 10$, $n(R1 \cap F \cap A) = 14$



i. $n(R \cap F^1 \cap A^1) = a + 2 + 3 + 7 = 14$

$$a + 12 = 14$$
$$a = 14 - 12 = 2$$

ii. $n(R^1 \cap F \cap A^1) = 2 + 7 + 7 + b = 18$

$$16 + b = 18$$
$$b = 18 - 16 = 2$$

iii. $n\left(R^1 \cap F^1 \cap A\right) = 3 + 7 + 7 + c = 23$

$$17 + c = 23$$
$$c = 23 - 17 = 6$$

iv. $n(R \cap F \cap A) = a + 2 + 7 + 7 + 3 + b + c$

$$= a + b + c + 19$$

But $a = 2, b = 2, and\ c = 6$

Therefore, $n(R \cap F \cap A) = 2 + 2 + 6 + 19 = 29$

Key ideas

- Venn diagrams are efficient and effective methods of solving application problems in sets (for example two set problems and three set problems).

Reflections

- How has the session exposed me to examples on applications of sets to effectively teach both two and three set problems in the classroom?

Discussions

1. In a first-year science class of Nsaba Presbyterian Senior High School, students were asked to make a choice between any one of the electives: chemistry and Geography or be left with no choice than to offer elective Math. 15 students opted for Chemistry as an elective and 27 opted for Geography. Given that one-third of the class declined the choices of Chemistry and Geography, find the number of students in the class.

2. In a survey of university students, 64 had taken mathematics course, 94 had taken chemistry course, 58 had taken physics course, 28 had taken mathematics and physics, 26 had taken mathematics and chemistry, 22 had taken chemistry and physics course and 14 had taken all three courses. How many had taken one course only?

3. In a survey of 150 people to determine their preference for these three automobiles: Mercedes, Nissan and Toyota, 90 people preferred Mercedes, 70

preferred Nissan and 80 preferred Toyota. 26 had preference for both Mercedes and Nissan, 30 preferred Mercedes and Toyota while 40 preferred Nissan and Toyota. Each one of the 150 had at least one preference. Represent the information on a Venn diagram and compute the number of people who preferred all 3 automobiles.

UNIT 3: RELATIONS, FUNCTIONS AND MAPPING

This unit introduces you to the concept of relations and functions, types of relations and the conditions for which a relation becomes a function. It also discusses equivalence and composition of functions. You will also learn about the domain, range as well as inverse of functions. Finally, the unit presents the algebra of functions, where you will learn how to add, subtract, multiply and divide two given functions.

Learning outcome(s)

By the end of the unit, the participant will be able to:

1. differentiate between relation and function;
2. determine the conditions for which a relation becomes a function;
3. determine the domain, range and inverse of functions;
4. determine equivalence relations;
5. find the composition of given functions;
6. add, subtract and multiply two or more functions.

SESSION 1: RELATIONS AND MAPPING

In this session, we shall learn about the meaning of relations and mapping; types of relations and under what conditions a relation becomes a function. The determination of domain, co-domain and range of a function will also be discussed.

Learning outcomes

By the end of the session, the participant will be able to:

a) define a relation and mapping;
b) state the four types of relations;
c) list the domain and the co-domain of ordered pairs;
d) distinguish between co-domain and range in terms of mapping.

Now read on …

Definitions of Relation and Mapping

A ***relation*** is a connection between two sets. That is, a relation associates the elements of one set to the elements of another set. A **mapping** is a kind of relation in which each member of the first set associates itself with a member in the second set. The set of all elements in the first set is called the ***domain*** and the set of all elements in the second set is called the ***co-domain***. Only the elements in the co-domain that are "used" by the relation constitute the ***range***. In other words, the range is the set of ***images*** of the elements of the domain. The range is a subset of the co-domain. ***Relation*** can also be defined as a set of ordered pairs. The first elements in the ordered pairs (the x-values), form the domain. The second elements in the ordered pairs (the y-values), form the range. The range is also called the dependent variable.

The mappings below shows a **relation** from set X to Y and set A into set B.

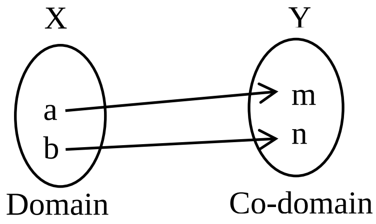


Figure 1: Relation from X to Y

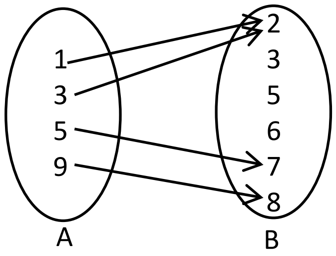


Figure 2: Relation from A to B

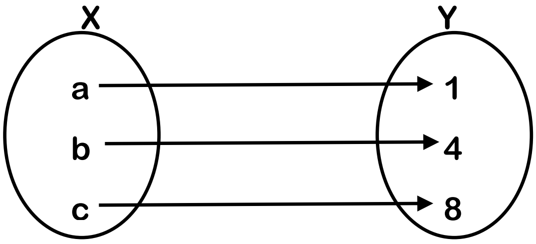
Now, look at the diagram in Figure 2 again. Take your pen and write down the domain, co-domain and the range of the relation.

Are your answers the same as these?
The **domain** is the set {1, 3, 5, 9}, the **co-domain** is the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and the **range** is the set {2, 7, 8}. Notice that 3, 5 and 6 are not part of the range because they are not images of any of the elements in the domain.

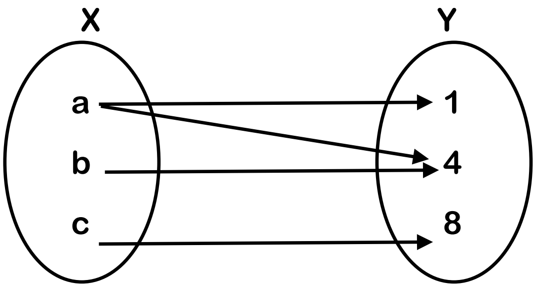
Can you write down the ordered pairs in Figure 2?
Is your answer the same as this? (1, 2), (3, 2), (5, 7), and (9, 8).

Types of Relations
There are basically four types of relations namely: **one-to-one, one-to-many, many-to-one,** and **many-to-many**. The arrow diagrams below illustrate the four types of relations.

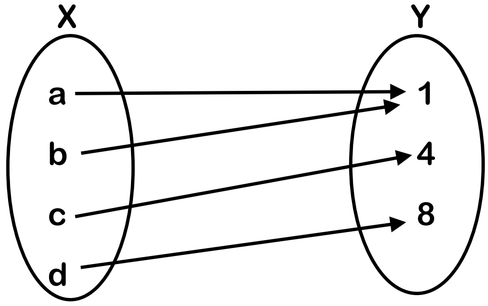
One-to-one relation: In this relation, each element in the domain has only one image in the co-domain and each element in the co-domain is associated with only one element in the domain. Thus, each element of the domain has a unique image.



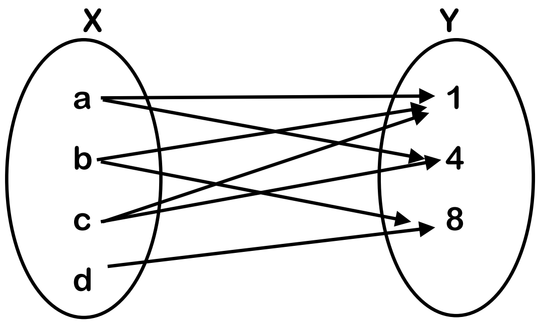
One-to-many relation: In this relation, one element in the domain is associated with many images in the co-domain.



Many-to-one relation: In this relation, several elements in the domain have one image in the co-domain.

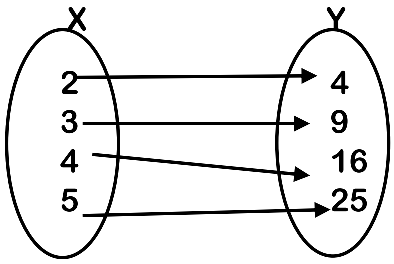


Many-to-many relation: In this relation, several elements in the domain have many images in the co-domain and several elements in the co-domain are associated with many elements in the domain.



Finding the rule for a mapping

The rule for a mapping is the relationship between the domain and the co-domain of the mapping. Consider the mapping of X into Y. If $y \in Y$ is the image of $x \in X$, then the rule for the mapping is given by $x \rightarrow y$. For example, we want to find the rule for the mapping below.



We can observe that each member in the domain is mapped into its square in the co-domain. That is: $2^2 = 4,\ 3^2 = 9,\ 4^2 = 16,\ etc.$ So if x stands for any member of the domain, then its image is x^2. The rule for the mapping is therefore $x \rightarrow x^2$.

Example 2: Find the rule for the following mapping

x	1	2	3	4
↓	↓	↓	↓	↓
y	4	8	12	16

Solution

It can be seen that each member is mapped onto a number four times itself in the co-domain. Hence, the rule for the mapping is $x \to 4x$

Example 3: Find the rule for the mapping below

x	1	2	3	4	5
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
y	8	10	12	14	16

Did you notice this about the elements in the co-domain?

x	1	2	3	4	5
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
y	2+6	4+6	6+6	8+6	10+6

A constant number (6) is added to twice each value of x. The rule for the maping is therefore, $x \to 2x + 6$.

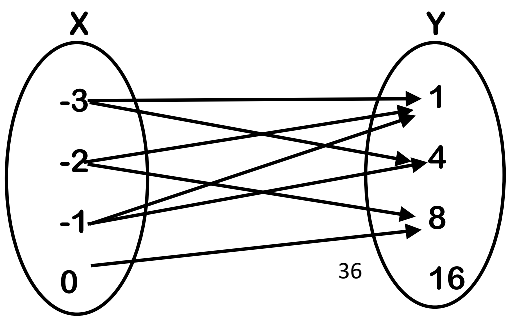
Key ideas

- A ***relation*** is a connection between two sets. That is, a relation associates the elements of one set to the elements of another set.
- A **mapping** is a kind of relation in which each member of the first set associates itself with a member in the second set.
- The set of all elements in the first set is called the ***domain*** and the set of all elements in the second set is called the ***co-domain***.
- The range is the set of ***images*** of the elements of the domain, also called the dependant variable.
- There are basically four types of relations namely: **one-to-one, one-to-many, many-to-one,** and **many-to-many**.

Reflections

- What are some of my experiences of teaching Relations and Mapping at the JHS level? How has the content of the session equipped me with the relevant information to effectively teach the concepts relations and mapping in a JHS classroom?

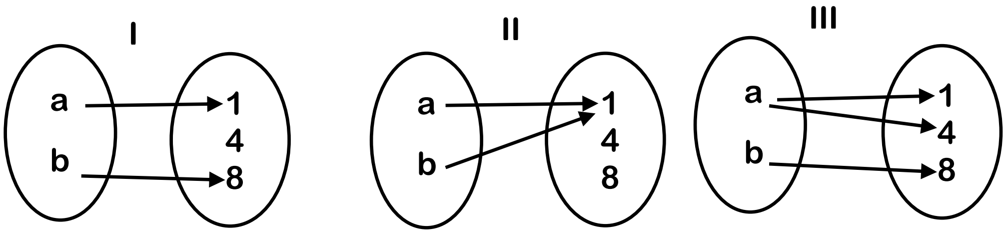
Discussions

1. (a) What do you understand by the term relation?
   (b) List and explain three types of relations.
2. 

(a) What is the range of the mapping?
(b) What is the rule of the mapping?

3. A function is defined by $y = 4x - 1$. List the range of the function over the domain $\{-2 \leq x \leq 2\}$

4. Which of the following is/are function(s)



5. Draw mapping diagram to show the image set of $\{1, 2, 4, 6, 8\}$ under the rule $x \rightarrow 3x^2 - 4$

SESSION 2: FUNCTIONS

In this session, we shall learn about functions and types of functions and the conditions for a relation to be a function.

Learning outcomes

By the end of the session, the participant will be able to:

a) define a function;
b) state the two types of functions;
c) prove that a function is one-to-one;
d) determine the domain and range of functions.

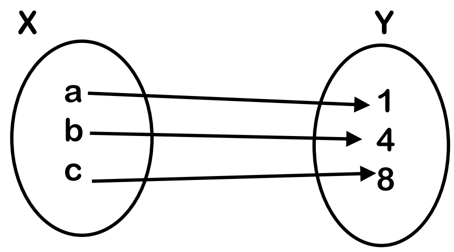
Now read on ...

Definition of a function

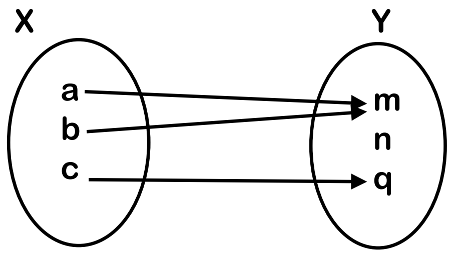
A function can be defined in several ways. For each definition of a function, there must be

(a) a set called the domain
(b) a set called the image and
(c) an association between the elements of the two sets, such that each element of the domain is paired with a unique element of the image set.

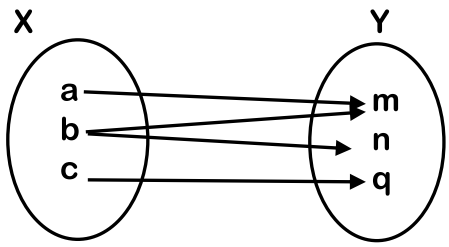
A function is a relation between two sets, say X and Y, such that each member of the set X is related to one and only one member of the set Y. That is, a relation in which each element of the domain has a unique image in the co-domain is a function.



This relation defines a function, since each element in X (domain) has just one image in Y (co-domain)



This relation also defines a function, because all elements in set X have only one image in set Y.



This relation does **not** define a function, since the element 'b' in the domain has more than one images m and q in the co-domain.

In terms of ordered pairs, a function is defined as a set of ordered pairs in which each *x*-element has only **one** *y*-element associated with it. In other words, a function is a set of ordered pairs in which no two different ordered pairs have the same first element. For example, let us consider the following ordered pairs {(1, 2), (2, 4), (3, 5), (2, 6), (1, -3)}. This set of ordered pairs is **not** a function because certain *x*-elements are paired with more than one unique *y*-element. That is, the element '2' in the domain has more than one image, 4 and 6, in the co-domain; (2, 4) and (2, 6).

Notice that **all** functions are relations but **not all** relations are functions.

Activity 1

Identify, with reasons, which of the following ordered pairs is/are functions

i) $A = \{(-3,1), (-1,1), (1,0), (3,0)\}$
ii) $B = \{(2,-2), (2,-1), (2,0), (2,1)\}$
iii) $C = \{(1,1)\}$
iv) $D = \{(1,0), (2,0), (3,0), (4,0)\}$

Check your answers:

Are your answers the same as these?

i) A is a function because each element, *x* has one and only one image, y
ii) B is **not** a function because the element 2 has more than one image, -2, -1, 0 and 1.
iii) C is a function
iv) D is a function because each element in the domain has a unique image in the range.

Note: In set language, a function may be defined as

$$f = \{(x, y) |\ x \in X, y \in Y, y = f(x)\}.$$

For instance, the relation "y is a function of x" is symbolised in many ways:

a) $f: x \to y$. Example $f: x \to 2x - 1$
b) $y = f(x)$. Example $y = 3x + 5$

Activity 2
A function is defined by the relation $f(x) = 5x^2 - 3$. Find the images of
(i) 3 (ii) -5

Check your answers with these:

Given $f(x) = 5x^2 - 3$.

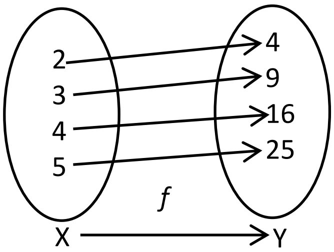
(i) Substitute $x = 3$ in the formula $f(x) = 5x^2 - 3$
$f(3) = 5(3)^2 - 3 = 5(9) - 3 = 42$

(ii) Substitute $x = -5$ in the formula $f(x) = 5x^2 - 3$
$f(-5) = 5(-5)^2 - 3 = 5(25) - 3 = 122$

Types of Functions

There are two main types of functions; one-to-one function and many-to-one function.

(i) One-to-one function: A function $f: X \to Y$ is a one-to-one if different elements in set X have distinct images in set Y.



Algebraically, we say a function is a one-to-one if $f(a) = f(b) \Rightarrow a = b$ for all $a, b \in R$, the set of real numbers.

Example 1: Show that the following functions are one-to-one

(i) $f: x \to x^2, for\ all\ x \geq 0$ (ii) $f: x \to \frac{1}{x+3}, x \neq -3$

(iii) $g(x) = \frac{1}{x^2-2}$, $x \neq 2$ (iv) $h(x) = \frac{x^2+4}{x^2}, x > 0$

Solution

(i) If *f* is a one-one function, then $f(a) = f(b) \Rightarrow a = b$
$\Rightarrow f(a) = f(b) = a^2 = b^2$
$\Rightarrow a = \pm\sqrt{b}$, but $a \geq 0 \Rightarrow a = b$.
Hence $f(a) = f(b) \Rightarrow a = b$, it implies the function *f* is one-to-one.

(ii) If *f* is a one-to-one function, then $f(a) = f(b) \Rightarrow a = b$
Thus $(a) = f(b) \Rightarrow \frac{1}{a+3} = \frac{1}{b+3}$. Cross multiplying gives

$$a + 3 = b + 3 \Rightarrow a = b$$

Since $f(a) = f(b) \Rightarrow a = b$, it follows that the function f is one-to-one.

(iii) For the function to be one-to-one, $g(a) = g(b) \Rightarrow a = b$

$g(a) = g(b) \Rightarrow \frac{1}{a^2-2} = \frac{1}{b^2-2}$. cross multiply

$\Rightarrow a^2 - 2 = b^2 - 2$

$\Rightarrow a^2 = b^2 \Rightarrow a = \pm\sqrt{b}$. thus $a = b$ or $a = -b$

$\therefore g(a) \neq g(b),$

Hence, the function g is **not** one-to-one

(iv) For the function to be one-to-one, $h(a) = h(b) \Rightarrow a = b$

$h(a) = h(b) \Rightarrow \frac{a^2+4}{a^2} = \frac{b^2+4}{b^2}$. cross multiply

$\Rightarrow a^2(b^2 + 4) = b^2(a^2 + 4)$. expanding gives

$\Rightarrow a^2b^2 + 4a^2 = b^2a^2 + 4b^2$

$\Rightarrow 4a^2 = 4b^2$

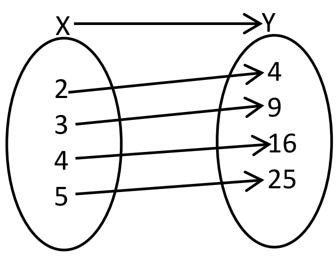
$\Rightarrow a^2 = b^2$

$\therefore a = b,$ since $a > 0.$

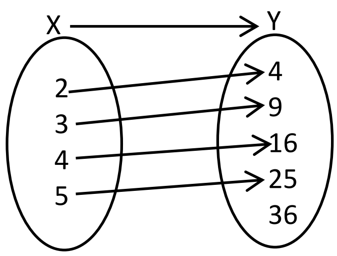
Hence, h is one-to-one.

(ii) Many-to-one function: This is another type of function where several elements in the domain have one image in the co-domain.

*Onto function***:** A function is said to be onto function if and only if the range of f is equal to the co-domain, otherwise it is an *into function*.



Onto function



Into function

Odd and Even Functions

Functions that are symmetrical about the vertical axis (y-axis) are called even functions. In terms of algebra, a function is ***even*** if for any value ‘’a’’, $f(a) = f(-a)$ or $f(-a) = f(a)$. Obvious examples of even functions are of the form $f(x) = x^n$, where n is an even integer, hence the name even function.

Example: Show that the function $f(x) = x^2 + 2$ is an even function.

Solution:

If the function is even, then $f(a) = f(-a)$

Given $f(x) = x^2 + 2$

$\Rightarrow f(a) = a^2 + 2$……………… (1)

Also, $f(-a) = (-a)^2 + 2$

$= a^2 + 2$…………… (2)

From (1) and (2), it can be seen that $f(a) = f(-a)$, hence the function is even.

A function with the property that $f(-a) = -f(a)$, for every member of the domain is called an ***Odd function***. Note that the graph of odd functions will have a rotational symmetry of order 2 about the origin (i.e. 180° rotation about the origin).

Example: Show that the function $f(x) = x^3 - 1$ is an odd function

Solution:
If the function is odd, then $f(-a) = -f(a)$
Given $f(x) = x^3 - 1$
⇨$f(-a) = (-a)^3 - 1 = -a^3 - 1 = -(a^3 + 1)$.......................... (1)
But $-f(a) = -[a^3 - 1] = -f(a)$
Since $f(-a) = -f(a)$, the function is odd.

Domain of a function

The domain of a function is the complete set of possible values of the independent variable (x). It is set of all possible x-values which will make the function "work", and the output real y-values.
Note: For a function to exist:

- the denominator (bottom) of a fraction must not be zero
- the number under a radical (square root) sign must be positive.

In general, we determine the domain by looking for those values of the independent variable (usually x) which will make the function defined. We have to also avoid 0 in the denominator of a fraction, or negative values under the square root sign.
For example, the function $f(x) = x^2 + 2$ is defined for all real values of x, because there are no restrictions on the value of x. Hence, the **domain** of $f(x)$ is "all real values of x".

Let us consider another function $y = \sqrt{x+4}$

The domain of this function is all real values of $x \geq -4$, since x cannot be less than −4. To see why, try out some numbers less than −4 (like −6 or −10) and some numbers more than −4 (like −3 or 5) in your calculator. The only ones that "work" and give us an answer are the ones greater than or equal to −4. This will make the number under the square root positive.

Example: Find the domain of the function $f(x) = \frac{x}{\sqrt{x-1}}$
Solution: Given that $f(x) = \frac{x}{\sqrt{x-1}}$, it can be seen that for the function to be defined $x - 1 > 0$. Solving this gives $x > 1$.
Hence, $Domain = \{x: x \in R, but\ x > 1\}$ or $Domain = \{x: x \in R, except\ x \leq 1\}$

Activity 1:
Find the domain for each of the following functions:
(a) $f(x) = x^2 + 2$ (b) $f(x) = \frac{x+5}{x-3}$ (c) $(x) = \frac{x}{\sqrt{25-x}}$ d) $f(x) = \sqrt{4 - x^2}$

Answers: (a) $Domain = \{x: x \in R\}$
(b) $Domain = \{x: x \in R, except\ x = 3\}$

(c) $Domain = \{x: x \in R, x < 25\}$
(d) $Domain = \{x: x \in R, -2 \leq x \leq 2\}$

Range of a function
The range of a function is the complete set of all possible resulting values of the dependent variable, after we have substituted the domain. The range is the resulting *y*-values we get after substituting all the possible *x*-values.

Tips for finding the range:

(i) Substitute different *x*-values into the expression for *y* to see what is happening. (Ask yourself: Is *y* always positive? always negative? Or maybe not equal to certain values?
(ii) Make sure you look for minimum and maximum values of *y*.
(iii) Draw a sketch. In mathematics, it's very true that a picture is worth a thousand words.

That is, to find the range of a function, first make x the subject and find the values of y which make x defined.

Example: State the range of the following functions:
1. $f: x \to 1 - 2x$ 2. $f: x \to \frac{1+x}{1-x}$ 3. $g: x \to \frac{1}{1+x^2}$

Solution

1. Let $f(x) = y \Rightarrow y = 1 - 2x$
Now make *x* the subject.
From $y = 1 - 2x$
$\Rightarrow 2x = 1 - y$
$\Rightarrow x = \frac{1-y}{2}$
The range is all possible values of y that will make x defined.
It is clear that all real numbers will make x 'work', therefore, the Range $= \{y: y \in R\}$

2. Let $f(x) = y \Rightarrow y = \frac{1+x}{1-x}$, make x the subject
$\Rightarrow y(1 - x) = 1 + x$
$\Rightarrow y - xy = 1 + x$
$\Rightarrow x = xy = y - 1$
$\Rightarrow x(1 + y) = y - 1$
$\therefore x = \frac{y-1}{1+y}$

It is obvious that y is defined for all R, except when $1 + y = 0 \Rightarrow y = -1$
Hence the range of $f = \{y: y \in R, except\ y = -1\}$ or $\{y: y \in R, y \neq -1\}$

3. Try to make *x* the subject. Check it, $x = \sqrt{\frac{1-y}{y}}$
You observe that x is defined between 0 and 1 with 1 inclusive.

Hence, the Range $= \{y; y \in R, 0 < y \leq 1\}$

Now take your pen and try the following example:
Find the domain and the range for the function g(s) $=\sqrt{3 - s}$

Compare your answer to this;
The domain is **not** defined for real numbers greater than 3, which would result in imaginary values for $g(s)$. Hence, the **domain** for $g(s)$ is "all real numbers, $s \leq 3$".
Also, by definition, g(s) = $\sqrt{3 - s}$ ≥ 0
Hence, the **range** of $g(s)$ is "all real numbers $g(s) \geq 0$"

Key ideas

- A function is a relation between two sets, say X and Y, such that each member of the set X is related to one and only one member of the set Y.
- In terms of ordered pairs, a function is defined as a set of ordered pairs in which each x-element has only **one** y-element associated with it.
- There are two main types of functions; one-to-one function and many -to-one function.
- In terms of algebra, a function is ***even*** if for any value ''a'', $f(a) = f(-a)$ or $f(-a) = f(a)$.
- A function with the property that $f(-a) = -f(a)$, for every member of the domain is called an ***Odd function***.

Reflections

- How has the content of the session equipped you with the relevant information to teach the concept of functions?

Discussions

1. A pupil argues that all relations are functions and all functions are relations. Explain how you would help the pupil to understand that this assertion is not always true.
2. Which of the following relations are functions?
   a) $\{(x, y) \mid y = 3x\}$ b) $\{(x, y) \mid y^2 = x\}$
3. Consider the function defined by $h(x) = 3x - 4$, for all R.
   (i) for what value of k is $h(k) = k$?
   (ii) for what values of x is $h(x) \geq x$?
4. (i) Distinguish between Odd and Even functions.
   (ii) Give an example each of an Odd and Even functions.

5. Find the domain and range for the following:
   (a) $f(x) = x^2 + 2$, for $x > 2$ (b) (b) $f(x) = x^3$, $-5 \leq x < 4$

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SESSION 3: EQUIVALENCE RELATIONS

The concepts of relations and functionsare used to solve problems in different topics in mathematics like probability, differentiation, integration, and so on. In this session, we shall discuss one other concept called "*equivalence relation"*.

Learning outcomes

By the end of the session, the participant will be able to:

a) define equivalence relations;
b) identify the properties of equivalence relations;

Now read on…….

A relation R on a set A is said to be an **equivalence relation** if and only if the relation R is reflexive, symmetric and transitive. We often use the notation a∼b to denote an equivalence relation. Note that since an equivalence relation is reflexive, it is automatically nonempty, provided set A is nonempty.

The relation "is equal to", denoted "=", is an equivalence relation on the set of real numbers since for any $x, y, z \in$ R:

1. $x = x$, (Reflexivity)

2. if $x = y$ then $y = x$, (Symmetry)

3. if $x = y$ and $y = z$ then $x = z$. (Transitivity)

All of these are true.

Reflexive: A relation is said to be reflexive, if (a, a) ∈ R, for every a ∈ A.
Symmetric: A relation is said to be symmetric, if (a, b) ∈ R, then (b, a) ∈ R.
Transitive: A relation is said to be transitive if (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R.
Equivalence relations can be explained in terms of the following examples:

- The sign '**is equal to**' on a set of numbers; for example, 1/3 is equal to 3/9.
- For a given set of triangles, the relation of '**is similar to**' and '**is congruent to**' all show equivalence.
- For a given set of integers, the relation of '**is congruent to**', modulo n' shows equivalence.
- The image and domain are the same under a function, shows the relation of equivalence.
- For a set of all angles, 'has the same cosine'.
- For a set of all real numbers,' has the same absolute value'.

Non-example: The relation "is less than or equal to", denoted "≤", is **not** an equivalence relation on the set of real numbers. For any x, y, z ∈ R, "≤" is reflexive and transitive but **not** necessarily symmetric.

Let us check it!
1. (Reflexivity): of course $x \le x$ is true since $x = x$.

2. (Symmetry): If $x \leq y$ then it is not necessarily true that $y \leq x$. For example, $5 \leq 7$, but $7 \leq 5$.
3. (Transitivity) If $x \leq y$ and $y \leq z$ then $x \leq z$ since $x \leq y \leq z$.

Proof of Equivalence Relation

Let us assume that R is a relation on the set of ordered pairs of positive integers such that ((a, b), (c, d)) ∈ R if and only if $ab = bc$. Is R an equivalence relation?

In order to prove that R is an equivalence relation, we must show that R is reflexive, symmetric and transitive.

Let us go through the proof as shown below

i) **Reflexive Property**
According to the reflexive property, if (a, a) ∈ R, for every a ∈ A. For all pairs of positive integers, ((a, b), (a, b)) ∈ R.
Clearly, we can say ab = ab for all positive integers.
Hence, the reflexive property holds.

ii) **Symmetric Property**
From the symmetric property, if (a, b) ∈ R, then we can say (b, a) ∈ R
For the given condition, if ((a, b), (c, d)) ∈ R, then ((c, d), (a, b)) ∈ R.
If ((a, b), (c, d)) ∈ R, then ad = bc and cb = da, since multiplication is commutative.
Therefore ((c, d), (a, b)) ∈ R
Hence symmetric property is proved.

iii) **Transitive Property**
From the transitive property, if (a, b) ∈ R and (b, c) ∈ R, then (a, c) also belongs to R
For the given set of ordered pairs of positive integers,
((a, b), (c, d)) ∈ R and ((c, d), (e, f)) ∈ R, then ((a, b), (e, f) ∈ R.
Now, assume that ((a, b), (c, d)) ∈ R and ((c, d), (e, f)) ∈ R.
Then we get, ad = cb and cf = de.
The above relation implies that a/b = c/d and that c/d = e/f, so a/b = e/f we get af = be.
Therefore ((a, b),(e, f))∈ R.
Hence transitive property is proved.

Example: Show that the relation R is an equivalence relation in the set
A = { 1, 2, 3, 4, 5 } given by the relation R = { (a, b):|a-b| is even}.

Solution: R = { (a, b):|a-b| is even }, where *a, b* belongs to set A

Reflexive Property:
From the given relation, |a – a| = | 0 |=0
And 0 is always even.
Thus, |a-a| is even
Therefore, (a, a) belongs to R
Hence R is Reflexive

Symmetric Property:

From the given relation, $|a – b| = |b – a|$
We know that $|a – b| = |-(b – a)|= |b – a|$
Hence $|a – b|$ is even,
Then $|b – a|$ is also even.
Therefore, if $(a, b) \in R$, then (b, a) belongs to R
Hence R is symmetric.

Transitive Property:
If $|a-b|$ is even, then $(a-b)$ is even.
Similarly, if $|b-c|$ is even, then $(b-c)$ is also even.
Sum of even numbers is also even
So, we can write it as $a-b + b-c$ is even
Then, $a – c$ is also even
So, $|a – b|$ and $|b – c|$ is even , then $|a-c|$ is even.
Therefore, if $(a, b) \in R$ and $(b, c) \in R$, then (a, c) also belongs to R
Hence R is transitive.

Key ideas

- A relation R on a set A is said to be an **equivalence relation** if and only if the relation R is reflexive, symmetric and transitive. That is, an **equivalence relation** is a **relation** that is reflexive, symmetric, and transitive.
- **Reflexive**: A relation is said to be reflexive, if $(a, a) \in R$, for every $a \in A$.
- **Symmetric**: A relation is said to be symmetric, if $(a, b) \in R$, then $(b, a) \in R$.
- **Transitive**: A relation is said to be transitive if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Reflections

- How has the content of the session equipped you with the relevant information to teach the concept of equivalence relations in the classroom?

Discussions

1) What is an equivalence relation?
2) State and explain the three properties of the equivalence relation?
3) Can we say the empty relation is an equivalence relation? Explain

SESSION 4: COMPOSITION OF FUNCTIONS

In the previous session, we have learnt about equivalence relations. In this session, we shall discuss composite functions and its properties.

Learning outcomes

By the end of the session, the participant will be able to:

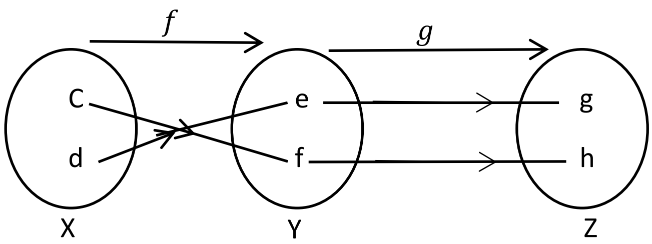
a) explain composition of functions

b) find the composition of two given functions
c) identify the properties of composite functions

Now read on …

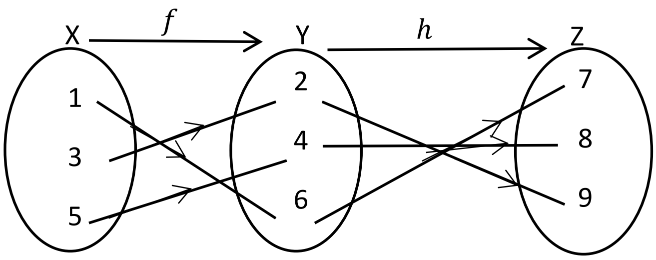
Composite functions

A composite function is the function composed into a single function, using separate functions in a defined order of operation. For instance, if we let $f: x \to y$ and $g: y \to z$, then we have the arrow diagram below.



The function which assigns each element of X into Z is called a **composite** function, defined as gof or gf (meaning *f* followed by *g*). Thus $gof(x) = g(f(x))$

Example 1: Given that $f: x \to Y$ and $h: Y \to Z$ are defined as shown below



Use the diagram above to answer the following: find

(i) $hof(1)$ (ii) $hof(3)$

Solution:

(i) $hof(1) = h[f(1)]$
$= h(6) = 7$

(ii) $hof(3) = h[f(3)]$
$= h(2) = 9$

Example 2: Given $f(x) = x + 1,\ g(x) = 5 - x$ and $h(x) = 2x^2$, find

(i) fog (ii) gof (iii) $fo(goh)$ (iv) $ho(gof)$

Solution:

(i) $fog = f[g(x)]$
$= f(5 - x)$
$= 5 - x + 1 = 6 - x$

(ii) $gof = g[f(x)]$

$= g(x+1) = 5-(x+1) = 4-x$

(iii) $goh = g[h(x)] = g(2x^2) = 5-x^2$

$\Rightarrow f[goh(x)] = f(5-x^2) = 5-2x^2+1 = 6-2x^2$

(iv) $gof = 4-x$

$\Rightarrow ho(gof) = h[gof(x)] = h(4-x)$

$= 2(4-x)^2 = 2(16-8x+x^2)$

$= 32-16x+2x^2$

Activity

If $f(x) = x-2$, $g(x) = 7-x$ and $h(x) = x^2+1$, find;

(i) $3(fog)$ (ii) $(fog)oh$ (iii) $ho(gof)$

Now compare your answers to these:

(i) $fog = f\big(g(x)\big) = f(7-x) = 7-x-2 = 5-x$

$\Rightarrow 3(fog) = 3(5-x) = 15-3x$

(ii) $fog[h(x)] = fog(x^2+1)$

$\Rightarrow (fog)oh = (5-x)[x^2+1]$

$= 5-(x^2+1) = 4-x^2$

(iii) $gof = g\big(f(x)\big) = g(x-2) = 7-(x-2) = 9-x$

$\therefore ho(gof) = h(9-x) = (9-x)^2+1$

$= 81-18x+x^2+1 = 82-18x+x^2$

Properties of Composite Functions

For composition of functions the following properties are observed;

1) Composition of functions is associative. For instance, given *f*, *g* and *h* as separate functions, $(fog)oh = fo(goh)$
2) Composition of function is distributive over addition. Given the separate functions f, g and h, $(f+g)oh = (foh)+(goh)$.

For example, if $f(x) = 2x+1$, $g(x) = x-1$ and $h(x) = x+1$,

$f+g = (2x+1)+(x-1) = 3x$.

$\therefore (f+g)oh = 3(x+1) = 3x+3$……………………………….(1)

Next, $foh = f\big(h(x)\big) = f(x+1) = 2(x+)+1 = 2x+3$

Also, $goh = g\big(h(x)\big) = g(x+1) = x+1-1 = x$

$\Rightarrow (foh)+(goh) = (2x+3)+x = 3x+3$……………………..(2)

Hence, $(f+g)oh = (foh)+(goh)$

3) Composition of functions is distributive over multiplication. If *f*, g and h are separate functions, $(f\cdot g)oh = (foh)\cdot(goh)$.

For example, using the functions above; $f(x) = 2x+1$, $g(x) = x-1$ and $h(x) = x+1$,

$f\cdot g = (2x+1)(x-1) = 2x^2-2x+x-1 = 2x^2-x-1$.

$\Rightarrow (f\cdot g)oh = (f\cdot g)(x+1) = 2(x+1)^2-(x+1)-1 = 2x^2+3x$

Now, $foh = f\big(h(x)\big) = f(x+1) = 2(x+1) + 1 = 2x + 3$.
Also, $goh = g\big(h(x)\big) = g(x+1) = (x+1) - 1 = x$
$\Rightarrow (foh) \cdot (goh) = (2x+3)(x) = 2x^2 + 3x$.
Hence, $(f \cdot g)oh = (foh) \cdot (goh)$

Key ideas

- A composition function is the function composed into a single function, using separate functions in a defined order of operation.

- Composition of functions is associative. For instance, given *f*, g and h as separate functions, $(fog)oh = fo(goh)$
- Composition of function is distributive over addition. Given the separate functions f, g and h, $(f+g)oh = (foh) + (goh)$.

- Composition of functions is distributive over multiplication. If *f*, g and h are separate functions, $(f \cdot g)oh = (foh) \cdot (goh)$.

Reflections

- How has the content of the session extended my experiences of teaching composite functions in the classroom?

Discussions

1. If $f(x) = 7x + 4$ and $g(x) = \frac{3x-1}{x-2}$, $x \neq 2$, find
   (i) fog (ii) gof (iii) $gof(2)$
2. Given $f: x \to 2 - x$, $g: x \to x + 1$, and $h: x \to 2x - 3$, verify the following:
   (a) $(fog)oh = fo(goh)$
   (b) $(f+g)oh = (foh) + (goh)$
   (c) $(f \cdot g)oh = (foh) \cdot (goh)$

3. Consider the functions: $h(x) = 2x + 1$, $g(x) = x^2 + 1$ and $f(x) = x - 1$
   Find: (i) $hofog$ (ii) $hofog(-3)$

SESSION 5: INVERSE OF FUNCTIONS

In the previous session, we learnt about composite functions and properties of composite functions. In this session, we shall learn about the inverse of functions.

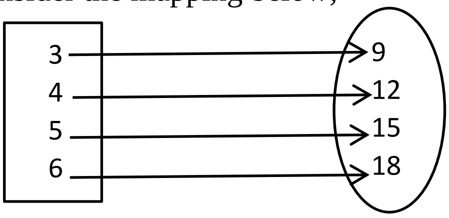
Learning outcomes

By the end of the session, the participant will be able to:

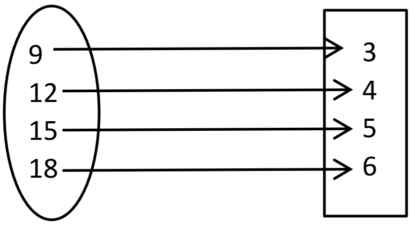
a) find the inverse of a function using arrow diagrams;
b) determine the inverse of a function graphically;
c) find the inverse of a function by calculation.

Now read on …

Inverse functions are functions by which mapping is from the range set to the domain set. Thus, elements of the range set rather map onto elements of the domain set. Let us consider the mapping below;



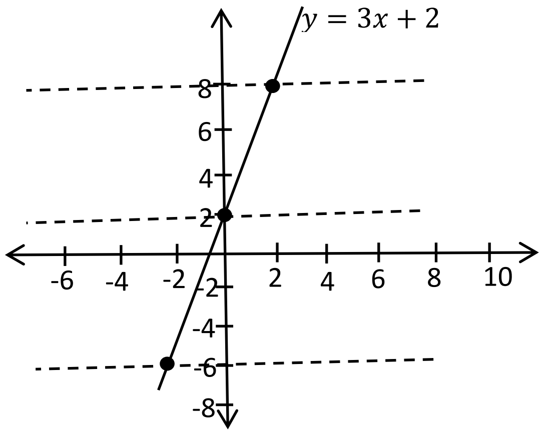
The above mapping depicts a function which can be denoted by $f: x \rightarrow 3x$. It is the mapping from the range to the domain.
Now, consider the following mapping



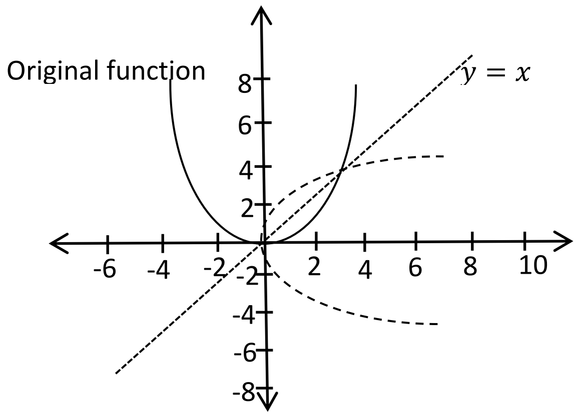
This mapping depicts a function that can be denoted by $f^{-1}: x \rightarrow \frac{1}{3}x$. The notation $f^{-1}: x$ is the inverse function of $f: x$. Hence the inverse of $f: x \rightarrow 3x$ is $f^{-1}: x \rightarrow \frac{1}{3}x$. Note the notation $f: x$ can be expressed as $f(x)$ and $f^{-1}: x$ can also be expressed as $f^{-1}(x)$.

How to determine graphically if a function has an inverse:

Use the ***horizontal line test*** to determine if a function has an inverse. If a horizontal line intersects your original function in ***only one*** location, your function has an inverse which is also a function. For example the function $y = 3x + 2$, as shown below has an inverse function because it passes the horizontal line test.



Another way to determine if a function has an inverse function is to find out if the reflection of the original function in the identity line, $y = x$, will also be a function (it passes the ***vertical line test*** for functions). For example, the graph of $y = x^2$ is shown below. The reflection of the graph over the identity line $y = x$ is shown with dashed line, its inverse relation. The dashed lines will not pass the vertical line test for functions, hence $y = x^2$ does not have an *inverse function*. You can see that the inverse relation exists, but it is **not** a function.



Note: With functions such as $= x^2$, it is possible to restrict the domain to obtain an inverse function for a portion of the graph. This means that only a selected section of the original graph will pass the horizontal line test for the existence of an inverse function.

Finding inverse by calculation

Example 1: If $f(x) = 5x + 2$, find the inverse, $f^{-1}(x)$

Solution:

Let $f(x) = y$

$\Rightarrow y = 5x + 2$,   make *x* the subject

$\Rightarrow 5x = y - 2$

$\therefore x = \frac{y-2}{5}$, interchanging y for x gives

Hence, the inverse of f is $f^{-1}(x) = \frac{x-2}{5}$

Example 2: Determine the inverse of the following functions:

1) $f(x) = 3 - 2x$ 2) $f(x) = \frac{2}{3}(x - 3)$

Solution

1) Given $f(x) = 3 - 2x,$ Let $f(x) = y$
That is $= 3 - 2x$, interchange x for y
That is $x = 3 - 2y$
Next make y the subject
$\Rightarrow 2y = 3 - x$
$\Rightarrow y = \frac{3-x}{2}$

Hence, the inverse, $f^{I}(x) = \frac{3-x}{2}$

2. Determine the inverse of $f(x) = \frac{2}{3}(x - 3)$

Solution

Let $f(x) = y$

$\Rightarrow y = \frac{2}{3}(x - 3)$, interchange the variables

$\Rightarrow x = \frac{2}{3}(y - 3)$, make y the subject

$\Rightarrow 3x = 2(y - 3)$

$\Rightarrow 3x = 2y - 6$

$\Rightarrow 2y = 3x + 6$

$\therefore y = \frac{3x+6}{2}$

Hence, the inverse of f is $f^{-1}(x) = \frac{3x+6}{2}$

Key ideas

- Inverse functions are functions by which mapping is from the range set to the domain set. Thus, elements of the range set rather map onto elements of the domain set.
- Horizontal line test and Vertical line test are two graphical approaches to determine if a function has an inverse.

Reflections

- How have the explanations and examples put forward in this session extended my experiences and knowledge to teach the idea of inverse functions in a JHS classroom?

Discussions

1) If $h(x) = \sqrt{4 - x^2}$, find the;
   (i) inverse of the function
   (ii) domain and the inverse of the function.
2) Determine the inverse of $f(x) = \frac{1}{x+1}$
3) Given that $f(x) = 5x - 2$ and $g(x) = 3x + 2$, where x is a real number, find
   (i) $g^{-1}(x)$
   (ii) $f\left(g^{-1}(x)\right)$

SESSION 6: ALGEBRA OF FUNCTIONS

In this session, we shall learn about the algebra of functions. We shall focus on the four basic operations: addition, subtraction multiplication and division.

Learning outcomes

By the end of the session, the participant will be able to:

a) find the sum of two given functions;
b) find the product of two given functions;
c) determine the difference of two functions;
d) divide two given functions ;
e) state the domain of the resulting function.

Now read on ……

Functions can be added, subtracted, multiplied and divided. Such procedures are referred to as "operations of functions" or "algebra of functions". These arithmetic procedures can be performed on two functions **when the functions have the same domains** (and no division by zero occurs). In other words, if two functions have a common domain, then arithmetic can be performed with them using the following definitions:

i. $(f + g)(x) = f(x) + g(x)$
ii. $(f - g)(x) = f(x) - g(x)$
iii. $(f \times g)(x) = f(x)g(x)$
iv. $\left(\frac{f}{g}\right)(x) = \frac{g(x)}{f(x)}$, where $g(x) \neq 0$

The domain for each of these new functions will be the intersection (∩) of the domains of functions *f* (*x*) and *g*(*x*). That is:

$$\left.\begin{array}{l} 1.\ D(f+g)(x) \\ 2.\ D(f-g)(x) \\ 3.\ D(f \times g)(x) \end{array}\right\} = D(f) \cap D(g)$$

4. $D\left(\frac{f}{g}\right)(x) = D(f) \cap D(g)$, where $g(x) \neq 0$

Example 1: Given that $f(x) = x + 4$ and $g(x) = x^2 - 2x - 3$, find each of the following and determine the common domain:

1. $(f + g)(x)$ 2. $(f - g)(x)$ 3. $(f \times g)(x)$ 4. $\left(\frac{f}{g}\right)(x)$

Solution:

1. $(f + g)(x)$=$(x + 4) + (x^2 - 2x - 3) = x^2 - x + 1$

The common domain is = $\{all\ real\ numbers\}$

2. $(f - g)(x) = (x + 4) - (x^2 - 2x - 3)$
$= x + 4 - x^2 + 2x + 3 = -x^2 + 3x + 7$

The common domain is $= \{all\ real\ numbers\}$

3. $(f \times g)(x) = (x + 4) \times (x^2 - 2x + 3)$
$= x^3 - 2x^2 - 3x + 4x^2 - 8x = -12$
$= x^3 + 2x^2 - 11x^2 - 12$

The common domain is = $\{all\ real\ numbers\}$

4. $\left(\frac{f}{g}\right)(x) = \frac{x+4}{x^2-2x-3} = \frac{x+4}{(x-3)(x+1)}$

This expression is undefined when x = 3 or when x = –1. So the common domain is {all real numbers except 3 or –1}.

Example 2: Given: $f(x) = 3x + 7$ and $(x) = \frac{3}{x-2}$, express $(f + g)(a)$ as a single fraction, and find $(f + g)(4)$

Solution:

$(f + g)(x) = f(x) + g(x) = 3x + 7 + \frac{3}{x-2}$

$$= \frac{(3x+7)(x-2)}{x-2} + \frac{3}{x-2}$$
$$= \frac{3x^2+x-14}{x-2} + \frac{3}{x-2} = \frac{3x^2+x-11}{x-2}$$

$(f + g)(4) = f(4) + g(4) = 19 + 3/2 = 20½$ (or $\frac{48+4-11}{4-2} =$ 20½)

Key ideas

- Functions can be added, subtracted, multiplied and divided through procedures referred to as "operations of functions" or "algebra of functions".
- Arithmetic procedures can be performed on two functions **when the functions have the same domains** (and no division by zero occurs).

Reflections

- How have the explanations and examples put forward in this session extended my experiences and knowledge to teach operations of functions in a JHS classroom?

Discussions

1) If $g(x) = 1 - x^2$ for all $x \in [-2, 2]$ and $h(x) = x^2$ for all $x \in R^+$, find:
(α) $(g + h)(x)$ (β) $(hg)(x)$

2) Consider the functions: $h(x) = 2x + 1$, $g(x) = x^2 + 1$ and $f(x) = x - 1$
Find:
i) the domain of $f(x) \cdot g(x)$
ii) $(f + g)\left(-\frac{2}{3}\right)$

3) A function f is defined on the set of real number by : $x \to \frac{x^2+1}{x^2-1}$, $x \neq \pm 1$. Find the value of x if $f: x \to \frac{10}{8}$

UNIT 4: LINEAR AND EXPONENTIAL SERIES

This unit introduces participants to some basic concepts in linear and exponential series and how to apply these concepts in solving real life problems.

Learning outcome(s)

By the end of the unit, the participant will be able to:
1. distinguish between linear and exponential sequences;
2. determine the terms of a recursively defined sequence;
3. find the arithmetic mean of a given linear sequence;
4. find the geometric mean of a given exponential sequence;
5. manipulate sequences other than *AP* or *GP;*
6. define a sequence using summation notation.

SESSION 1: SEQUENCES AND SERIES

You are warmly welcome to the first session of Unit 4. This session focusses on the concepts of Arithmetic (Linear) sequence and Geometric (Exponential) sequence.

Learning outcomes
By the end of the session, the participant will be able to:
a) distinguish between sequences and series;
b) distinguish between linear and exponential sequences and series;
c) find the general term of linear sequences;
d) write recursive formula for arithmetic sequence;
e) find the general term of exponential sequences;
f) write recursive formula for geometric sequence;
g) solve problems related to recursive sequence.

Now read on…

Definitions of Sequence and Series
Study each of the following numbers and determine the next four terms.

(i) 2, 4, 6, 8, __, __, __, __
(ii) 2, 5, 8, __, __, __, __
(iii) 1, 2, 4, 8, 16, __, __, __, __

Compare your answer to these: (i) 10, 12, 14, 16; (ii) 11, 14, 17, 20 (iii) 32, 64, 128, 256.
Observe that the list of numbers in (i), (ii) and (iii) follows a specific order. A **sequence** is formed when a list of numbers is presented in a specific order. For example, 2, 4, 6, 8, 10, 12, 14, 16 … is a sequence. Thus, a sequence is a set of numbers each of which can be obtained from the preceding one by a definite law. Each number or expression forming the set is called a *term* of the sequence. A **series**

is formed by forming the sum of the terms of a sequence. For example, 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16… is a series. That is, when the terms of a sequence are added a series is formed.

Now study each of the following examples and find the next four terms.

(i) 1, 3, 6, 10, 15, __, __, __, __.

(ii) 1, 4, 7, 10, __, __, __, __.

(iii) 2, 4, 8, 16, __, __, __, __.

(iv) 4, 2, 1, ½, ¼, __, __, __, __.

(v) 2, 6, 12, 20, 30, __, __, __, __

Linear sequence

Any sequence in which successive terms increase (or decrease) by a constant is called a **linear sequence** or an **arithmetic progression (AP)**. An *AP* is a sequence in which each term is derived from the previous term by adding a constant. The constant is called *common difference*, (*d*); it can either be positive or negative. Consider the sequence: $2, 5, 8, \_, \_, \_, \_$. Each succeeding term is obtained by adding 3 (common difference) to the preceding term. Each term can also be written in terms of the first term, a and the common difference, d.

The first term, $U_1 = a = 2$

The second term, $U_2 = 2 + 1(3) = 5$

The third term, $U_3 = 2 + 2(3) = 8$

The fourth term, $U_4 = 2 + 3(3) = 11$

In general, an *AP* with first term, a, and common difference, d, has an nth term, $U_n = a + (n-1)d$ or $U_n = U_1 + (n-1)d$.

Example 1

For each of the following linear sequences (*AP*), determine the $8th$ term and the nth term.

(i) $16, 14, 12, 10, ...$ (ii) $6, 2, -2, -6,$

Solution

(i) $16, 14, 12, 10, ...$

The first term, $a = 16$ and the common difference, $d = 14 - 16 = -2$

The 8th term is $U_8 = a + (8-1)d = a + 7d == 16 + 7(-2) = 2$

The nth term is obtained as $U_n = a + (n-1)d$

$= 16 + (n-1)(-2) = 18 - 2n$

(ii) $6, 2, -2, -6, ...$

The first term, $a = 6$ and the common difference, $d = 2 - 6 = -4$

The 8th term is $U_8 = a + (8-1)d = a + 7d = 6 + 7(-4) = -22$

The nth term therefore, is $U_n = a + (n-1)d$

$= 6 + (n-1)(-4) = 10 - 4n$

Example 2

The 3rd term of an *AP* is -1 and the 10th term is 20. Find the:

(a) 20th term (b) general nth term of the sequence

Solution

(a) The 3rd term = -1, $\Rightarrow a + 2d = -1 \ldots\ldots\ldots (1)$

Also, the 10th term = 20, $\Rightarrow a + 9d = 20 \ldots\ldots\ldots. (2)$

Solving the two equations gives $a = -7$ and $d = 3$
Hence, the first term is -7 and the common difference is 3

Using $Un = a + (n - 1)d$, the 20th term will be
Therefore, the 20th term, $U_{20} = -7 + (20 - 1)(3) = 50$

(b) The general *n*th term, $U_n = a + (n - 1)d \quad \Rightarrow U_n - 7 + 3n - 3 = 3n - 10$

Example 3
A 5th term of an *AP* is thrice the 2nd term. If the first term is 8, find the:
(a) nth term of the sequence (b) 10th term of the sequence.

Solution
Given, $U_5 = 3(U_2)$, implies $a + 4d = 3(a + d)$
If $a = 8$, implies $a + 4d = 3(a + d)$
$d = 2a = 2(8) = 16$
Then $U_n = 16n - 18$
And $U_{10} = 16(10) - 8 = 152$

Recursive Sequence
Sequences such as (i) $-2, 2, 6, 10, \ldots, 4n - 6$ and (ii) $5, 7, 9, 11, \ldots, 2n + 3$ follow an explicit formula because they allow direct computation for any term of the given sequence. You do not need to determine the term prior to the first term in order to figure out what the *n*th term is going to be. A sequence is recursively defined if the first term is given and there is a method of determining the *n*th term by using the terms that precede it. A recursive formula is a formula that calculates each term of a sequence based on the preceding term (term that came right before). They will have an U_{n-1} somewhere in the equation which represents the term prior to the term you are solving for, and they will give you a specific value.

- Explicit formula: $U_n = a + (n - 1)d$
- Recursive formula: $U_n = U_{n-1} + d$.

Recursive formula for Arithmetic Sequence
$U_n = U_{n-1} + 3$ is an example of a recursive formula for arithmetic sequence with common difference, $d = 3$. It means that take the prior term and just add 3 to find the next.

Example 1: Write the recursive formula for the sequence: 6, 8, 10, 12, …

Solution: The sequence is arithmetic sequence with the common difference 2. Therefore, the recursive formula can be written as $U_n = U_{n-1} + 2, U_1 = 6$

Example 2: Write the recursive formula for the sequence 1, -3, -7, -11, …

Solution: The sequence is arithmetic sequence with the common difference -4. Therefore, the recursive formula can be written as $U_n = U_{n-1} - 4, U_1 = 1$

Example 3: Find the 6th term: $\boldsymbol{U_n = U_{n-1} + 5}$ where $\boldsymbol{U_1 = 2}$

Solution

Given $U_n = U_{n-1} + 5$, $U_1 = 2$

$U_2 = U_1 + 5 = 2 + 5 = 7$

$U_3 = U_2 + 5 = 7 + 5 = 12$

$U_4 = U_3 + 5 = 12 + 5 = 17$

$U_5 = U_4 + 5 = 17 + 5 = 22$

$U_6 = U_5 + 5 = 22 + 5 = 27$

Therefore the 6^{th} term is 27

Exponential sequence

An **exponential sequence** or **geometric progression** *(GP)* is a sequence in which each term is a constant multiple of the preceding term. Considering the sequence 2, 6, 18, 54, 162, -, -, -, we can see that each succeeding term can be obtained by multiplying the preceding term by a constant number, 3. The constant multiplier is called the *common ratio*. Also, we can write each term in terms of the first term, 2 and the common ratio, 3.

$$U_1 = 2 = 2(3^0)$$
$$U_2 = 6 = 2(3^1)$$
$$U_3 = 18 = 2(3)^2$$
$$U_4 = 54 = 2(3)^3$$
$$U_5 = 162 = 2(3)^4$$
$$\dots \dots \dots.$$
$$U_n = ar^{n-1}$$

An exponential sequence is generally of the form: a, ar, ar^2, ar^3, …, ar^{n-1}, where a is the first term and r, the common ratio. The general or the n^{th} term of a *GP* is given by $U_n = ar^{n-1}$

If U_1, U_2, U_3, U_4, -, -, -, are the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} terms respectively of a sequence, then the common ratio, $r = \frac{U_2}{U_1} = \frac{U_3}{U_2} = \frac{U_4}{U_3}$

Example 1: Find the 15^{th} term of the *GP* 3, 6, 12, …

Solution: $a = 3$, $r = \frac{6}{3} = \frac{12}{6} = 2$

But $U_n = ar^{n-1} \Rightarrow U_{15} = 3(2)^{15-1} = 3(2)^{14} = 49{,}152$

Example 2: Find the n^{th} term of the sequence whose first term is 36 and 4^{th} term is $4\frac{1}{2}$.

Solution: $a = 36, \quad U_4 = 4\,\frac{1}{2}$

$\Rightarrow ar^3 = 4\frac{1}{2}$

But $a = 36 \Rightarrow 36r^3 = 4\frac{1}{2}$

$\Rightarrow r^3 = \frac{1}{8} \Rightarrow r = \frac{1}{2}$

Now, $U_n = ar^{n-1} = 36\left(\frac{1}{2}\right)^{n-1} = 36\left(\frac{1}{2}\right)^{n} \times 2 = 72\left(\frac{1}{2}\right)^{n}$

Therefore, the n^{th} is $72\left(\frac{1}{2}\right)^{n}$.

Example 3: Find the number of terms in the exponential sequence $1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{243}$

Solution: $a = 1, r = \frac{1}{3}$ and $U_n = \frac{1}{243}$

Using, $U_n = ar^{n-1} \Rightarrow \frac{1}{243} = 1\left(\frac{1}{3}\right)^{n-1}$

$$\Rightarrow \frac{1}{243} = \left(\frac{1}{3}\right)^{n-1}$$

$$\Rightarrow \frac{1}{3^5} = \left(\frac{1}{3}\right)^{n-1} \Rightarrow \left(\frac{1}{3}\right)^{3} = \left(\frac{1}{3}\right)^{n-1}$$

$$\Rightarrow n - 1 = 5 \qquad \therefore\ n = 6$$

Hence there are 6 terms in the sequence.

Recursive formula for Geometric Sequences

For example, $U_n = 0.5U_{n-1}$ is a recursive formula for geometric sequence with common ratio, $r = 0.5.$ It means that take the prior term and just multiply by 0.5 to find the next term. Thus, recursive formula for geometric sequence means that take the previous term and multiply by common ratio, *r* to continue the pattern.

Example 1: Write the recursive formula for the sequence: 3, 6, 12, 24…

Solution: This is a geometric sequence with common ratio 2. Thus, the recursive formula is given by $U_n = 2U_{n-1}, U_1 = 3$

Example 2: Find the 16th term $U_n = 3.5U_{n-1}, U_{15} = 21$

Solution: Given $U_n = 3.5U_{n-1}, U_{15} = 21$
$U_{16} = 3.5U_{15} = 3.5 \times 21 = 73.5$

Problems Related to Recursive Sequence

The rules for recursive formulas may look more complicated but are actually very logical and they are often easier to use than the explicit formula.

Example 1: Given that $U_1 = 1,\quad U_n = nU_{n-1}$, find U_1, U_2, U_3, U_4 and U_5. Hence, describe the sequence of U_n.

Solution: The first term is given as U_1 = 1. To get the second term, we use $n =$ 2 and the formula $U_n = nU_{n-1}$ to get $U_2 = 2U_1 = 2.1 = 2.$ To get the third term, we use $n = 3$ in the value of the preceding term. Thus,

$$U_1 = 1$$
$$U_2 = 2.1 = 2$$
$$U_3 = 3.2 = 6$$
$$U_4 = 4.6 = 24$$
$$U_5 = 5.24 = 120$$

Hence, U_n can be described as *n* factorial, $U_n = n!$.

Example 2: Write down the first five terms of the following recursively defined sequence:

$u_n = 1, u_2 = 1$, If $u_{n+2} = u_n + u_{n+1}$

Solution: We are given the first two terms. To get the third term requires that we know both of the previous two terms. That is, $u_1 = 1, u_2 = 1$

$$u_3 = u_1 + u_2 = 1 + 1 = 2$$
$$u_4 = u_2 + u_3 = 1 + 2 = 3$$
$$u_5 = u_3 + u_4 = 2 + 3 = 5$$

The sequence defined in Example 2 is called the **Fibonacci sequence**, and the terms of this sequence are called **Fibonacci numbers**.

Key ideas

- A sequence is formed when a list of numbers is presented in a specific order.
- A sequence in which successive terms increase (or decrease) by a constant is called a *linear sequence* or an *arithmetic progression* (AP).
- An *AP* is a sequence in which each term is derived from the previous term by adding a constant. The constant can either be positive or negative and it is called *common difference*, d.
- In general, an AP with first term, a and common difference, d, has an nth term, $U_n = a + (n-1)d$ or $U_n = U_1 + (n-1)d$.
- Exponential sequence or geometric progression *(GP)* is a sequence in which each term is a constant multiple of the preceding term. The constant multiplier is called the *common ratio*.
- An exponential sequence is generally of the form: a, ar, ar^2, ar^3, …, ar^{n-1}, where a is the first term and r, the common ratio. The general or the n^{th} term of a *GP* is given by $U_n = ar^{n-1}$
- If U_1, U_2, U_3, U_4, -, -, -, are the 1st, 2nd, 3rd and 4th terms respectively of a sequence then the common ratio, $r = \frac{U_2}{U_1} = \frac{U_3}{U_2} = \frac{U_4}{U_3}$
- A sequence is recursively defined if the first term is given and there is a method of determining the *n*th term by using the terms that precede it.
- A recursive formula is a formula that calculates each term of a sequence based on the preceding term (term that came right before).
- Recursive formulas can be used for arithmetic or geometric sequences.
- $U_n = U_{n-1} + 3$ is an example of a recursive formula for arithmetic sequence with common difference, $d = 3$. It means that take the prior term and just add 3 to find the next.
- An example of recursive formula for geometric sequence with common ratio, $r = 0.5$., $U_n = 0.5U_{n-1}$. It means that take the prior term and just multiply by 0.5 to find the next term.

Reflections

- How have the explanations and examples put forward in this session extended my experiences and knowledge to teach linear and geometric sequences in a high school?
- How has the content of this session broadened your understanding on recursive sequence to effectively teach the concept in a JHS classroom?

Discussions

1) Find the 12th term of the sequence 8, $7\frac{1}{4}, 6\frac{1}{2}$, ...
2) Find the number of terms in the sequence 2, 7, 12, …,42
3) A car depreciates by 20% each year. The cost of the car when bought is GH¢30,000. Calculate the value of the car after the fifth year.

4) Find the 11th term of the sequence with the first term $1\frac{1}{2}$ and common difference $\frac{-7}{4}$.
5) The 5th and 10th terms of a linear sequence are -12 and -27 respectively. Find the sequence and its 15th term.
6) The consecutive terms of an *AP* have the sum 15 and product 80. Find the numbers.

7. Mr. Motey's rent increased by GH¢60 every year. If in 20 years, he paid a total of GH¢21,400 as rent, find his rent:
   a) for the first year;
   b) in the 20th year.
8. (a) Write the recursive formula for the sequence 9, 1, -7, -15…
   (b) Write the explicit formula to find the 30th term. Hence, determine the 30th term.
9. (a) Write the first 5 terms of the sequence using the explicit formula given, $U_n = 2n + 10$.
   (b) Then, write the recursive formula for the sequence.
10. A sequence of numbers U_1, U_2, U_3, …. satisfies the relation $(3n – 2)\, U_{n+1} = (3n + 1)U_n$ for all positive integers n. If $U_1 = 1$, find:
   a) U_3 and U_4 b) the expression for U_n

SESSION 2: SUM OF THE TERMS OF AP AND GP

This session focuses on the sum of indicated terms of Linear sequences and Exponential sequences.

Learning outcomes

By the end of the session, the participant will be able to:
i. find the sum of indicated terms of linear sequences;
ii. find the sum of indicated terms of exponential sequences;
iii. solve problems related to real life.

Now read on …

The Sum of the first *n* terms of an *AP*
Consider the sum of the first *n*th terms of the AP with the following terms,

$$a, a + d, a + 2d, a + 3d, \dots, + a + (n - 1)d$$

Summing up the sequence, we have

$$Sn = a + (a + d) + (a + 2d) + (a + 3d) + l \dots\dots\dots\dots..(1)$$

where l is the last term and it is given as $l = a + (n - 1)d$
Reversing equation (1), we have

$$Sn = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a \dots\dots\dots..(2)$$

Adding equation (1) and (2), we have

$$2Sn = (a + l) + (a + l) + (a + l) + \cdots$$

$$2S_n = n(a + l)$$

$$S_n = \frac{n}{2}(a + l)$$

But $l = a + (n - 1)d$, therefore,

$$S_n = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$$

Hence the sum of the first n terms of an AP is

$$S_n = \frac{n}{2}[2a + (n - 1)d] \text{ or } S_n = \frac{n}{2}(a + l)$$

Example 1: Find the sum of the first 10 terms of the sequence.

(a) $-7, -4, -1, 2 \ldots$ (b) $0, -½, -1, \frac{-3}{2}, \ldots$ (c) $2, 1\ 1/2, \ldots$

Solution

(a) $a = -7$, $d = 3$, $S_n = \frac{n}{2}[2a + (n - 1)d]$

$$S_{10} = \frac{10}{2}[2(-7) + (10 - 1)3]$$
$$= 5(-14 + 27) = 65$$

(b) $a = 0$, and $d = (-\frac{1}{2})$, $S_n = \frac{n}{2}[2a + (n - 1)d]$

$$S_{10} = \frac{10}{2}\left[2(0) + (10 - 1)(-\frac{1}{2})\right] = -22.5$$

(c) $a = 2$, and $d = -½$

$$S_{10} = \frac{10}{2}\left[2(2) + (10 - 1)(-\frac{1}{2})\right] = -2.5$$

Example 2: The sum of the first 5 terms of a sequence is 30 and the sum of the first 4 terms is 20. Find the 5th term.

Solution: Given $S_5 = 30$, and $S_4 = 20$,

$$U_n = S_n - S_{n-1}$$
$$U_5 = S_5 - S_{5-1}$$
$$U_5 = S_5 - S_4$$
$$U_5 = 30 - 20 = 10$$

Therefore, the fifth term is 10.

Example 3: In an *AP*, the 6th term is thrice the 2nd term and the 9th term is 27. Find

(a) sum of the 1st *n* terms; (b) the sum of the first 20 terms.

Solution

(a) $U_6 = 3(U_2) \Rightarrow a + 5d = 3(a + d) = 3a + 3d$

$\Rightarrow a = d$ ………………….. (1)

Also, $U_9 = 27 \quad \Rightarrow a + 8d = 27$ ………… … (2)

Substituting (1) into (2) gives

$d + 8d = 27 \Rightarrow d = 3$ and $a = 3$ (since $a = d$)

Hence, the first term, $a = 3$ and the common difference, $d = 3$.

Now, $Sn = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[2(3) + (n - 1)3]$

$$= \frac{n}{2}(6 + 3n - 3) = \frac{n}{2}(3n + 3)$$

(b) The sum of the first *n* terms is $S_n = \frac{3n}{2}(n + 1)$

Then, $S_{20} = \frac{3(20)}{2}(20 + 1) = 30(21) = 630$

Hence the sum of the first 20 terms is 630.

Example 4: Mr. Asare's salary started with GH¢24,000 and increased by annual increment of GH¢2,000 to a maximum of GH36, 000.00. After how many years did he obtain the maximum salary? How much would Mr. Asare earn altogether, after 10 years in the job?

Solution: The first term, $a = 24{,}000$, common difference, $d = 2000$ and $Un = 36000$. Substituting these into $U_n = a + (n - 1)\,d$

$$\Rightarrow 36000 = 24000 + (n - 1)2000$$

$$\Rightarrow n = 7 \text{ years}$$

Hence, Mr. Asare had his maximum salary after 7 years.

Since the sequence breaks down after 7 years, from the 8th year he will receive a fixed amount of GH¢36000 per year for 3 years.

So, the amount for the 3 years = $3 \times 36{,}000 = 108{,}000$

For the first 7 years he got, $S_7 = \frac{7}{2}[\,2\,(24{,}000) + (6)2000] = 210{,}000$

So, for ten years, he would have = $210{,}000 + 108{,}000 = GH¢318{,}000.00$

The sum of the first *n* terms of *GP*

Recall that the first *n* term of an exponential sequence are a, ar, ar^2, ar^3, - , -, -, ar^{n-1}.
Let S_n represents the sum of the sequence. Thus

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots\ldots\ldots\ldots + ar^{n-1} \quad \ldots\ldots\ldots\ldots\ (1)$$

Multiplying equation (1) by *r*, we have

$$rS_n = ar + ar^2 + ar^3 + ar^5 \ldots\ldots\ldots\ldots\ldots + ar^n \ldots\ldots\ldots\ldots\ (2)$$

Now, equation (1) – (2) gives $S_n - rS_n = a - ar^n$

$$\Rightarrow S_n = \frac{a(1 - r^n)}{1-r},\ r < 1$$

Alternatively, (2) – (1) we, have $rS_n - S_n = ar^n + a$

$$\Rightarrow (r - 1)S_n = a(r^n - r)$$

$$\therefore S_n = \frac{a(r^n - 1)}{r-1}$$

Thus, the sum of first *n* terms of a *GP* is given by

$S_n = \frac{a(r^n-1)}{r-1}$, for $r > 1$ or, $S_n = \frac{a(1-r^n)}{1-r}$, for $r < 1$

Example 1: Find the sum of the first five terms of the sequence 1, 3, 9, 27, …

Solution: $a = 1,\ r = 3$, Since $r > 1$, we use $S_n = \frac{a(r^n-1)}{r-1}$

$$\Rightarrow S_5 = \frac{1[(3)^5 - 1]}{3-1} = \frac{242}{2} = 121$$

Example 2: The 7th term of a *GP* is 56 and the 4th term is 7. Find the sum of the first 10 terms of the sequence.

Solution

$U_7 = 56 \Rightarrow ar^6 = 56$ ……………. (1)

Also, $U_4 = 7 \Rightarrow ar^3 = 7$ ………….. (2)

(1) ÷ (2) $\Rightarrow \frac{ar^6}{ar^3} = \frac{56}{7} \Rightarrow r^2 = 8$

$$\therefore r = 2$$

Put $r = 2$ into (2) $ar^3 = 7$

$\Rightarrow a\,(2)^3 = 7$

$8a = 7 \Rightarrow a = \frac{7}{8}$

$\therefore$ Using $S_n = \frac{a(r^n-1)}{r-1} \Rightarrow S_{10} = \frac{7}{8}\frac{[2^{10}-1]}{2-1} = \frac{7}{8}\,(1023) = 895.125$

Example 3: A mother bought 100 kg of rice for the family. Each week, she cooked one-tenth of the rice left over from the previous week. Find the:

i) total quantity of the rice cooked by the end of the 10th week;

ii) number of weeks she took to cook 90 kg of rice.

Solution:

1st Week	2nd Week	3rd Week	4th Week………
↓	↓	↓	↓
10	9	$\frac{81}{10}$	$\frac{729}{100}$……………

$\Rightarrow a = 10$ and $r = \frac{9}{10}$

i) $S_n = \frac{10\left(1-\left(\frac{9}{10}\right)^{10}\right)}{1-\frac{9}{10}} = 65.132$ kg

ii) $90 = \frac{10\left(1-\left(\frac{9}{10}\right)^{n}\right)}{1-\frac{9}{10}}$

This gives $\left(\frac{9}{10}\right)^n = \frac{1}{10} \Rightarrow (0.9)^n = 0.1$

Taking the logarithm of both sides to base 10 gives $n\log 0.9 = \log 0.1$

And $n = 21.85 \approx 22$

Therefore, it will take 22 weeks to cook 90 kg of rice.

The sum to infinity of a GP

The sum to infinity is given by $S_\infty = \frac{a}{1-r}$.

Example 1: Write down the sum to infinity of the following series.

i) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \dots\dots\dots\dots$

ii) $12 + 6 + 3 + 1\frac{1}{3} + \cdots \dots\dots..$

iii) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots ....$

Solution

i) $S_n = \frac{a(r^n-1)}{r-1}$

$= \frac{1\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$, but $\left(\frac{1}{3}\right)^\infty = 0$

$= S\infty = \frac{3}{2}(1) = \frac{3}{2}$

ii) $S_n = \frac{12\left(1-\left(\frac{1}{2}\right)^n\right)}{1/2} = 24\left(1-\left(\frac{1}{2}\right)^n\right)$, but $\left(\frac{1}{2}\right)^\infty = 0$

$S\infty = 24(1) = 24$

iii) $S_n = \frac{1\left(1-\left(-\frac{1}{2}\right)^n\right)}{1/2} = 1\left(1-\left(-\frac{1}{2}\right)^n\right)$, but $\left(-\frac{1}{2}\right)^\infty = 0$

$S\infty = 1(1) = 1$

Recurring Decimals

Recurrent decimals are repeated decimals. Every recurrent decimal represents a rational number. For example, $\frac{1}{3} = 0.33333$………………or $0.\dot{3}$

$$\frac{2}{11} = 0.181818 \ldots\ldots\ldots\ldots\ldots\ldots or\ 0.\dot{1}\dot{8}$$

Every recurrent decimal can be expressed as quotient $\left(\frac{r}{q}\right)$ of two integers by first writing it as an infinite exponential series, with first term *a*, and common ratio, *r*. For example,

(a) $\frac{1}{3} = 0.3333 \Rightarrow 0.3 + 0.3 + 0.003 + \cdots \ldots ....$

$$\Rightarrow = \frac{3}{10} + \frac{3}{10^2} + \frac{18}{100^3} + \ldots\ldots$$

(b) $\frac{8}{15} = 0.53333 = 0.5 + 0.03 + 0.003 + \cdots$

$$= \frac{5}{10} + \frac{3}{10^2} + \frac{3}{10^2} + \cdots \ldots\ldots\ldots\ldots\ldots$$

Example 1: Express $0.\dot{6}$ as an infinite geometric series and hence find the sum of the series.

Solution: $0.\dot{6}$ = 0.6+ 0.06 + 0.006 +……….. $= \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \ldots..$

Hence 0.6 is an infinite series of first term, $\frac{6}{10}$ and a common ratio, $\frac{1}{10}$.

Since $|r| < 1$, the sum to infinite

$$\Rightarrow 0.\dot{6} = \frac{a}{1-r} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{2}{3}$$

Example 2: Express $0.1\dot{6}$ in the form $\frac{p}{q}$, where *p* and *q* are integers.

Solution: $0.1\dot{6}$ = 0.16666 = 0.1 + 0.06 + 0.006 + 0.00006 + ……

$$= \frac{1}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \ldots\ldots..$$

But $\frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \cdots \ldots\ldots$ is a GP with $a = \frac{6}{10^2}$ and $r = \frac{1}{10}$

Thus, $\frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \ldots\ldots = \frac{\frac{6}{100}}{1-\frac{1}{10}} = \frac{1}{15}$

$$\therefore 0.1\dot{6} = \frac{1}{10} + \frac{1}{15} = \frac{1}{6}$$

Key ideas

- The idea of linear sequence can be used to solve problems related to real life situations.
- The idea and the principles of the geometric progression can be used to solve problems related to real life.

Reflections

- How has the session equipped me with the relevant examples and approaches to teach application of linear sequence and exponential sequence in the classroom?

Discussions

1. The sum of three consecutive terms of an *AP* is 18, and their products is 120. Find the terms.
2. The 5th term of a linear sequence is 12 and the 12th term is 25. Find the:
   (a) sum of the first 30 terms,

(b) sum of the terms from 5^{th} to 15^{th} inclusive.

3. In an arithmetic progression, the 8^{th} term is twice the 4^{th} term and the 20^{th} term is 40. Find the sum of the terms from the 8th to the 20th inclusive.
4. A child building a tower with blocks uses 15 for the bottom row. Each row has 2 fewer blocks than the previous row. Suppose that there are 8 rows in the tower.
   a) How many blocks are used for the top row?
   b) What is the total number of blocks in the tower?
5. The 5^{th} term of a *GP* is 162 and the 8^{th} term is 4374. Find the sum of the first 10 terms
6. Express the following recurring decimals in the form $\frac{p}{p}$
   i) $0.1\ddot{3}6$ ii) $0.41\dot{6}$ iii) $0.4\ddot{1}3$
7. If the sum of infinity of a GP is three times the first term. What is the common ratio?
8. The sum of *n* terms of a certain series is $4^n - 1$ for all values of *n*. Find the first three terms and the n^{th} term and show that the series is a *GP*.
9. The consecutive odd numbers *p, q* and *r* are such that the sum of *p* and *r* is 34. Find *p, q* and *r*.
10. In an experimental sequence, the 6^{th} term is 8 times the 3^{rd} term and the sum of the 7^{th} and 8^{th} terms is 192. Find the sum of the 5^{th} to 11^{th} terms inclusive.
11. A man starts saving on 1^{st} April. He saves one pound the first day two pounds the second day, four pounds the third day, and so on. Doubling the amount every day. If he managed to keep on saving under this system until the end of the month (30 days), how much would he save? Give your answer in pounds, correct to three significance figures.
12. The sum of the first *n* terms of a series is $2n^2 - n$. Find the *n*th term and show that the series is an *AP*.

SESSION 3: ARITHMETIC MEAN AND GEOMETRIC MEAN

This session deals with the concepts of arithmetic means and geometric means of given sequences.

Learning outcomes

By the end of this session, participants will be able to

1. find an arithmetic mean of two given numbers in a sequence;
2. find an geometric mean of two given numbers in a sequence.

Now read on …

Arithmetic Mean

If three numbers a, b, c are consecutive terms of an *AP*, then *b* is called, the *arithmetic mean* of a and c. Thus, if a, b, c are in *AP* then the common difference is therefore $b - a$ or $c - b$

$$\Rightarrow\ b - a = c - b$$
$$\Rightarrow\ 2b = c + a$$
$$\therefore\ b = \frac{c + a}{2}$$

$\therefore$ The arithmetic mean of a and c is $\frac{a+c}{2}$**.** That is average of a and c.

Example 1: Find the arithmetic mean of 4 and 64.
Solution: Arithmetic Mean = $\frac{4+64}{2} = \frac{68}{2} = 34$.
Example 2: Insert 3 arithmetic means between 24 and 8.
Solution: Let the means be x, y, z. Then $24, x, y, z, 8$ forms an *AP*.
Thus, $a = 24$, and $a + 4d = 8 \Rightarrow 24 + 4d = 8 \Rightarrow d = -4$

$$x = a + d = 24 - 4 = 20$$
$$y = a + 2d = 24 + 2(-4) = 16$$
$$z = a + 3d = 24 + 3(-4) = 12$$

The required arithmetic means are 20, 16, 12.

Geometric Mean
If a, b, c are consecutive terms of a *GP*, then b is called the *geometric mean* of a and c
The common ratio, r is $\frac{b}{a}$ or $\frac{c}{b}$

$$\Rightarrow \frac{b}{a} = \frac{c}{b}$$
$$\Rightarrow b^2 = ac$$
$$\therefore b = \sqrt{ac}$$

Example 1: Find the geometric mean of 4 and 64.
Solution: Geometric Mean = $\sqrt{(4 \times 64} = \sqrt{256} = 16$

Example 2: Insert two geometric means between 4 and 32.
Solution: Let the means be x and y. then 4, x, y, 32 for a *GP*.
Therefore, $a = 4$ and $ar^3 = 32 \Rightarrow r = 2$

$$x = ar = 4 \times 2 = 8$$
$$y = ar^2 = 4 \times 2^2 = 16$$

Therefore, the required means are 8 and 16

Key ideas

- If three numbers a, b, c are consecutive terms of an *AP*, then b is called, the *arithmetic mean* of a and c. it is given by $b = \frac{c+a}{2}$
- If a, b, c are consecutive terms of a *GP*, then b is called the *geometric mean* of a and c and can be deduced as $b = \sqrt{ac}$

Reflections

- How have the explanations and examples put forward in this session extended my experiences and knowledge to teach arithmetic and geometric means in a high school?

Discussions

1. Insert 5 arithmetic means between 12 and 21.
2. Insert 6 arithmetic means between 12 and 25.
3. Insert 3 arithmetic means between 8 and 18.
4. Insert 4 geometric means between 5 and 1215.
5. Insert three geometric means between $2^1/_4$ and $^4/_9$.

SESSION 4: SEQUENCE OTHER THAN *AP* OR *GP*

Let us consider the sequence 1, 5, 13, 22, 33, 46, …, U_n. You could see that this sequence is neither an *AP* nor a *GP* because you cannot determine the common difference or the common ratio respectively. This session focuses on a sequence which is neither an *AP* nor a *GP*. The session also covers the use of sigma notation in representing series.

Learning outcomes

By the end of the session, the participant will be able to:

a) determine the general or *n*th term (U_n) of a sequence which is neither an *AP* nor a *GP*,
b) use sigma notation in representing series,
c) apply the use of sigma notation in finding sum.

Now read on…….

Sequence neither *AP* nor *GP*

Sometimes you may come across a sequence which is neither an *AP* nor a *GP* for example;

1, 5, 13, 22, 33, 46, …, U_n.

In such instance, we need to find a way of determining the general term, U_n of such sequence. Let us consider the sequence U_1, U_2, U_3, …, U_n.

1. If the first difference of the sequence is an *AP*, then the n^{th} term takes the form of a quadratic, i.e. $U_n = an^2 + bn + c$, where *a, b* and *c* are constants.
2. If the second difference of the sequence is an *AP*, then the n^{th} term takes the form a cubic i.e. $U_n = an^3 + bn^2 + cn + d$, where *a, b, c* and *d* are constants.
3. If the second difference is a *GP*, then the n^{th} term takes a power function of the form $U_n = a^n + n$, where *a* is a constant.

Example 1: Find the general term of the sequence and hence determine value for 12^{th} term.

0, 3, 8, 15, 24, 35, …

Solution: The sequence 0, 3, 8, 15, 24, 35, … is neither an *AP* nor a *GP*. So, let us find the first difference.

U_n	1^{st} Difference
0	
	3
3	
	5
8	
	7
15	
	9
24	
	11
35	

We can see that the first difference: 3, 5, 7, 9, 11 is an *AP*, hence the sequence, U_n is in a quadratic form. We will use $u_n = an^2 + bn + c$ to determine the general nth term of the sequence.

When $n = 1$, we have $U_1 = a + b + c = 0 \ldots\ldots\ldots\ldots.(1)$
When $n = 2$, we have $U_2 = 4a + 2b + c = 3 \ldots\ldots\ldots\ldots.(2)$
When $n = 3$, we have $U_3 = 9a + 3b + c = 8 \ldots\ldots\ldots\ldots.(3)$
Solving equations (1), (2), and (3) simultaneously, we have;
(2) - (1): $3a + b = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots...(4)$
(3) – (2): $5a + b = 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$
(5) – (4): $2a = 2$
Thus, $a = 1$, $b = 0$ and $c = -1$
Therefore, the general nth term of the sequence is $U_n = n^2 - 1$.
The 12th term will be $U_{12} = 12^2 - 1 = 144 - 1 = 143$

Example 2: Find the general term of the sequence and hence determine value for 8th term. 0, 7, 26, 63, 123, …
Solution: The sequence 0, 7, 26, 63, 123, … is neither an *AP* nor a *GP*.

U_n	1st Diff.	2nd Diff.
0		
	7	
7		12
	19	
26		18
	37	
63		23
	60	
123		

It is the second difference that gave us an *AP*. So, the sequence U_n is of the form $an^3 + bn^2 + cn + d$, where a, b, c and d are constants.
When $n = 1$, we have $U_1 = a + b + c + d = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$
When $n = 2$, we have $U_2 = 8a + 4b + 2c + d = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$
When $n = 3$, we have $U_1 = 27a + 9b + 3c + d = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$
When $n = 4$, we have $U_1 = 64a + 16b + 4c + d = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$
Solving equation (1), (2), (3), and (4) simultaneously, we have $a = 1$, $b = 0$, $c = 0$, and $d = -1$.
Therefore, the general nth term of the sequence is $U_n = n^3 - 1$.
The 8th term will be $U_n = 8^3 - 1 = 512 - 1 = 511$

Example 3: Find the general term of the sequence $3, 6, 11, 20, 37, \ldots$
Solution: The sequence $3, 6, 11, 20, 37$ ... is neither an *AP* nor a *GP*
Sequence of first differences: 3, 5, 9, 17, …..
Sequence of second differences: 2, 4, 8, …..
The second difference is a GP. So, the sequence is of the form $U_n = a^n + n$, where a is a constant.
When $n = 1$, we have $a + 1 = 3 \Rightarrow a = 2$.
When $n = 2$, we have $a^2 + 2 = 6 \Rightarrow a^2 = 4$ Thus $a = \pm 2$
When $a = 2$, we have the sequence to be: 3, 6, 11, 20, 37, …
When $a = -2$, we have the sequence to be -1, 6, -7, 20, -27,… which is a different sequence. Therefore, the general term for the sequence 3, 6, 11, 20, 37, ….is $U_n = 2^n + n$.

Summation Notation

It is often important to be able to find the sum of the first *n* terms of a sequence $\{a_n\}$, that is,

$$a_1 + a_2 + a_3 + \cdots \ldots\ldots\ldots. + a_n$$

Rather than writing down all the terms, summation notation, $\sum$ can be used to provide a more concise way of expressing the sum. Summation notation is also called *sigma notation*. The symbol $\sum$ is called sigma (Greek letter). The notation is used to represent both finite sums and infinite sums. Let us consider the sum of the first *n* terms of a sequence, $\{a_n\}$ as

$$a_1 + a_2 + a_3 + \cdots \ldots\ldots\ldots. + a_n.$$

The summation notation for this sequence can be written as $\sum_{(k=1)}^{n} a_k$.

That is, $a_1 + a_2 + a_3 + \ldots\ldots. + a_n = \sum_{k=1}^{n} a_k$

The notation includes a lower bound (e. g. $k = 1$) and an upper bound ($k = n$) that together identify the terms over which the sum occurs. The integer *k* is called the index of the sum, it tells you where to start the sum and where to end it. Thus, the expression, $\sum_{k=1}^{n} a_k$, means add the terms a_k of the sequence $\{a_n\}$ starting with $k = 1$ and ending with $k = n$. We read the expression $\sum_{k=1}^{n} a_k$ as "the sum of a_k from $k = 1$ *to* $k = n$".

Properties of Sum of Sequence

If $\{a_n\}$ and $\{b_n\}$ are two sequences and *c* is a real number, then:

(1) $\sum_{k=1}^{n}(ca_k) = ca_1 + ca_2 + \ldots\ldots\ldots.. + ca_n = c(a_1 + a_2 + \ldots + a_n) = c\sum_{k=1}^{n} a_k$

(2) $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(3) $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(4) $\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k$, where $0 < j < n$.

Formulas for Sum of Sequences

(1) $\displaystyle\sum_{k=1}^{n} c = c + c + \cdots \ldots\ldots\ldots + c = cn,\quad c\ is$ a real number

(2) $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots \ldots\ldots\ldots. + n = \frac{n(n+1)}{2}$

(3) $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots \ldots\ldots \frac{n(n+1)(2n+1)}{6}$

(4) $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots. + n^3 = \frac{n^2(n+1)^2}{4}$

Example 1: Express each sum using summation notation

a) $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots.. + 9^2$ $\quad$ $b)\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots.. + \frac{1}{2^{n-1}}$

Solution

a) The sum $1^2 + 2^2 + 3^3 + \ldots\ldots\ldots + 9^2$ has 9 terms, each of the form k^2, and start at $k = 1$ and ends at $k = 9$.
Hence, $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots. + 9^2 = \sum_{k=1}^{9} k^2$

b) The sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \ldots. + \frac{1}{2^{n-1}}$ has n times, each of the form
Hence, $\frac{1}{2^{n-1}} = \sum_{k=1}^{n} \frac{1}{2^{k-1}}$

The index of summation needs not always begins at 1 or end in n; for example, we could have expressed the sum we obtained for (b) as $\sum_{k=0}^{n-a} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} \ldots\ldots\ldots.. + \frac{1}{2^{n-a}}$ each represents the same sum as the one given in the example above

Example 2: Write out each sum in full.

$$a) \sum_{k=1}^{n} \frac{1}{k} \qquad\qquad b) \sum_{k=1}^{n} k!$$

Solution

a) $\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ \ldots\ldots\ldots.. + \frac{1}{n}$

b) $\sum_{k-1}^{n} k! = 1! + 2! + \cdots \ldots. + n!$

Example 3: Find the sum of each of the following sequences.

$$\text{(a)} \sum_{k-1}^{5}(3k) \qquad \text{(b)} \sum_{k=1}^{10}(k^2 + 1) \qquad \text{(c)} \sum_{k=1}^{24}(k^2 - 7k + 2) \qquad \text{(d)} \sum_{k=6}^{20}(4k^2)$$

Solution

(a) $\sum_{k=1}^{5}(3k) = 3\sum_{k=1}^{5} k = 3\left(\frac{5(5+1)}{2}\right) = 3(15) = 45$

$$\text{(b)} \sum_{k=1}^{10}(k^2) + \sum_{k1}^{10} 1 = \frac{10(10+1)(2(10)+1)}{6} + 1(10) = 385 + 10 = 395$$

$$(c) \sum_{k=1}^{24}(k^2 - 7k + 2) = \sum_{k=1}^{24} k^2 - \sum_{k=1}^{24} 7k + \sum_{k=1}^{24} 2 = \sum_{k=1}^{24} k^2 - 7\sum_{k=1}^{24} k + \sum_{k=1}^{24} 2$$

$$= \frac{24(24+1)[2(24)+1]}{6} - 7\left(\frac{24(24+1)}{2}\right) + 2(24) = 2848$$

(d) Note that the index of summation starts at 6. We use property (4) as follows

$$\sum_{k=6}^{20}(4k^2) = 4\sum_{k=6}^{20} k^2$$

$$= 4[\sum_{k=1}^{20} k^2 - \sum_{k=1}^{5} k^2] \ = 4\ \left[\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right] = 11260$$

Key ideas

- Some sequences are neither an Arithmetic Progression (AP) nor a Geometric Progression (GP). For example, 1, 5, 13, 22, 33, 46, …, U_n.
- Summation notation, $\sum$ can be used to provide a more concise way of expressing the sum of a sequence. The notation is used to represent both finite sums and infinite sums.

Reflections

What are some of my experiences in handling sequences that are neither an AP nor a GP? How has the session prepared me to effectively teach the concept and its applications in a JHS classroom?

Discussions

1. Find the general term of each of the following sequences:
   (i) -2, 1, 5, 13, 22, 33, …
   (ii) 8, 15, 26, 41, 60, …
   (iii) 7, 14, 33, 70, 131, …
   (iv) 4, 11, 30, 85, 248, …
2. Determine the sum S_n of the first n terms of the sequence,
   8 + 11 + 14 + …+ ($3n$ +5).
3. Find the sum of the first n terms of a series 3 + 7 + 13 + 23 + 41 + …+ ($2^n + 2n - 1$).

UNIT 5: CONVERGENCE AND DIVERGENCE SERIES

This unit focuses on intuitive treatment of convergence and divergence series, P-series and Harmonic series. Special treatment of various methods used to determine convergence and divergence of series, particularly Comparison test, Root test and Ratio test are also discussed.

Learning outcome(s)

By the end of the unit, the participant will be able to:

1. intuitively determine convergence or divergence of series;
2. identify p-series and harmonic series;
3. conduct the comparison test to determine convergence or divergence of series;
4. use the root test to determine convergence or divergence of series; and
5. carry out the ratio test to determine convergence or divergence of series.

SESSION 1: INTUITIVE TREATMENT OF CONVERGENCE AND DIVERGENCE SERIES

In this session, we shall discuss the intuitive treatment of convergence and how to intuitively determine convergence or divergence of series. We will also take a look at conditions under which an exponential series converges or diverges

Learning outcomes

By the end of the session, the participant will be able to:

a) state the intuitive definition of convergence or divergence of series; and
b) state the conditions under which a given exponential series converges or diverges;
c) determine whether an exponential series converges or diverges;

Now read on …

Intuitive treatment of convergence and divergence of series

Consider the infinite series $u_1 + u_2 + u_3 + \ldots + u_r + \ldots$

The series $u_1 + u_2 + u_3 + \ldots + u_r + \ldots$ is said to converge (or is convergent) if there is a number, $L=0$, such that

$$\lim_{n\to\infty} S_n = \sum_{r=1}^{\infty} u_r = L.$$

The number L is called the sum of the infinite series. If there is such number other than 0, then the series is said to diverge (or divergent).

On the other hand, let us consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Now if

$$\lim_{n \to \infty} S_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

If there is no such number, then the series is said to diverge (or is divergent).
The sum of the 1st, 2nd, 3rd 4th, 5th , 6th and 7th terms of the series (to 2 decimal places) gives:

1, 1.5, 1.83, 2.08, 2.28, 2.45, 2.59, . . .

It can be seen that each successive sum increases without bound as *n* gets bigger and bigger, we say the series diverges or is divergent. It is also possible to have some series whose sum doesn't get closer to a particular value but oscillates indefinitely between two values. Such a series is said to diverge (or is divergent). An example of such a series is

$$s_n = \sum_{n=1}^{\infty} (-1)^n$$

You will notice that the sum, s_n is 0 when *n* is even and -1 when *n* is odd. Since the sum oscillates between these two values, the series is said to diverge.
Think about other series that behave in similar fashion.

Example 1: Explain why the series $\sum_{n=1}^{\infty} 2n$ diverges.

Solution: $\sum_{n=1}^{\infty} 2n = 2 + 2 + 2 + 2 +$ ...

The series diverges to ∞ since its nth partial sum is $S_n = 2n \neq 0$

Example 2: Explain why the series $\sum_{n=1}^{\infty} (-1)^{n+1}$ is divergent.

Solution: $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

The partial sum of the series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

This oscillates between 1 and 0 for *'n'* odd and even respectively, then the series is divergent.

Note: Wrong use of algebra can mislead you to drawing wrong conclusions. Can you think of two ways by which this can happen?

Recall that the sum of the exponential series is given as

$$S_n = \frac{a(1-r^n)}{1-r}, for\ |r| < 1 \text{ and } S_n = \frac{a(r^n-1)}{r-1}, |r| > 1.$$

Now for $r = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} < 1$, let us examine what happens to r^n as n gets bigger and bigger.

$r^n = \frac{1}{3^n}$ for n = 1, 2, 3, 5, 10, 40, 100, 200, 500, *etc.*

You may use your calculators and write down your responses for discussion.
Now, write down what happens to r^n.
You might have noticed that as n gets bigger and bigger, r^n turns to 0. If this is your observation excellent and keep it up.

Now if we replace $r^n = 0$; that is, using very large values of n ;

$$S_n = \frac{a(1-r^n)}{1-r},$$

we have $S_\infty = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

Thus sum to infinity $S_\infty = \frac{a}{1-r}$, for $|r| < 1$.

Take some time to examine situations where $|r| > 1$. If you noticed that, r^n gets bigger and bigger, thumps up for you. In this case, can we say there is sum to infinity? From your observation, notice that

1. For the case of $|r| < 1$, there is a sum to a particular value represented by $S_\infty = \frac{a}{1-r}$, and
2. For the case of $|r| > 1$, there is no such definite sum

We can conclude that, when **|r| < 1 the geometric series converges to** $\boldsymbol{\frac{a}{1-r}}$ (i.e. a converging sum of $\frac{a}{1-r}$) and **diverges if** $\boldsymbol{|r|} > 1$. This observation is used to determine whether a exponential series converges with a converging sum or diverges with no such sum.

Example 3: Is the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent or divergent? If it is convergent, what is the converging sum?

Solution: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

You will notice that the series is an exponential series with $r = \frac{1}{2}$. Can you explain why $r = \frac{1}{2}$?

Since |r|= $\frac{1}{2}$ < 1, the series convergent.

The converging sum is: $S_\infty = \frac{a}{1-r}$,

From the series the value of $a = \frac{1}{2}$

$S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$. Therefore the converging sum is 1.

Example 4: Does the series $\sum_{n=1}^{\infty} \frac{1}{(-3)^{n-1}}$ converge?

Solution: $\sum_{n=1}^{\infty} \frac{1}{(-3)^{n-1}} = 1 + \frac{1}{-3} + \frac{1}{9} + \frac{1}{-27} + \frac{1}{81} + \frac{1}{-243} + \cdots$

This is an infinite exponential series with first term $a = 1$ and a common ratio $r = -\frac{1}{3}$.

Now the |r|= $\left|-\frac{1}{3}\right| = \frac{1}{3}$ < 1, hence the series converges to $\frac{a}{1-r} = \frac{1}{1-(\frac{1}{3})} = \frac{1}{1-(-\frac{1}{3})} = \frac{3}{4}$

Example 5: Explain why the series $\sum_{n=1}^{\infty} \frac{1}{(-3)^{n-1}}$ diverges.

Solution: Expanding the series gives: $\sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + \cdots$

This is an exponential series with first term 2 and the ratio 2.

Since the $|r| = |2| = 2 > 1$, the series diverges.

Key ideas

- When **$|\mathbf{r}| < 1$ the geometric series converges to $\boldsymbol{\frac{a}{1-r}}$** (i.e. a converging sum of $\frac{a}{1-r}$) and **diverges if $|\boldsymbol{r}| > 1$.** This observation is used to determine whether a an exponential series converges with a converging sum or diverges with no such sum.
- The series $u_1 + u_2 + u_3 + \ldots + u_r + \ldots$ is said to converge (or is convergent) if there is a number, $L=0$, such that

$$\lim_{n\to\infty} S_n = \sum_{r=1}^{\infty} u_r = L.$$

- The number L is called the sum of the infinite series. If there is such number other than 0, then the series is said to diverge (or divergent).

Reflections

- How has the ideas shared in this session prepared me to teach convergence and divergence series?
- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently handle a lesson on the topic "sum to infinity" in a mathematics classroom?

Discussions

1. State the intuitive definition of convergence or divergence of series
2. Explain why each of the following series diverges:

   $a)$ $\sum_{\mathrm{n}=1}^{\infty} -4\mathrm{n}$ $\qquad$ $b)$ $\sum_{\mathrm{n}=1}^{\infty} (-2)^{\mathrm{n}+1}$

3. Show that the series $\sum_{x=1}^{\infty} \frac{x}{x+1}$ diverges.
4. Determine whether the series converges and find the convergent sum if possible of the following:

   a) $\sum_{n=1}^{\infty} \frac{1}{(5)^{n-1}}$ $\qquad$ b) $\sum_{n=1}^{\infty} \frac{1}{3^n}$

4. Explain why the series $\sum_{n=1}^{\infty} -3^n$ diverges

SESSION 2: P-SERIES AND HARMONIC SERIES

In this session, our discussion is on convergence and divergence of a slight variation to the exponential series called the P-Series and Harmonic Series.

Learning outcomes

By the end of the session, the participant will be able to:

a) identify what P-series and Harmonic series are, and
b) determine whether a given P-series or Harmonic series converges or diverges.

Now read on …

P-Series and Harmonic Series

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p} + ...$ is called the **p-series**.

Its sum is finite for p > 1 (i.e. converges) and is infinite for p ≤1 (i.e. diverges).
If p = 1, we have the **harmonic series**.

Note: The p-series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + ..$ is a *standard series* known as ***harmonic series*** which is **divergent.**

Example 1: Explain why the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Solution: Expanding gives $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Observe that $p = 2 > 1$. Since $p > 1$ the series converges.

Example 2: Show that the series $\sum_{n=1}^{\infty} \frac{2}{3n}$ diverges.

Solution: We can rewrite the series as $\sum_{n=1}^{\infty} \frac{2}{3n} = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n} = \frac{2}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$

Does the series in the bracket look familiar? What kind of series is that?

This is identified as a harmonic series. Since the harmonic series diverges, a product of it will also diverge.

Hence the series $\sum_{n=1}^{\infty} \frac{2}{3n}$ $diverges$

Key ideas

- The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p} + ...$ is called the p-series. Its sum is finite for p > 1 (i.e. converges) and is infinite for p ≤1 (i.e. diverges). If p = 1, we have harmonic series.

- The p-series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + ..$ is a *standard series* known as ***harmonic series*** which is **divergent.**

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently handle a lesson on the topic "P-Series and Harmonic Series" in a mathematics classroom?

Discussions

1. With the use of an example explain why a given p-series is convergent or divergent?
2. What is Harmonic series?

SESSION 3: COMPARISON TEST

This session takes participants through processes leading to determining convergence or divergence of series by the comparison test.

Learning outcomes
By the end of the session, the participant will be able to:

a) state the conditions by which a given series converges or diverges using the comparison test;
b) use the comparison test to determine the convergence or divergence of a given series.

Now read on …

Comparison Test
The convergence or divergence of a given series may be determined by comparing its terms with terms of another series that is known to be convergent or divergent. Such a test is called *comparison test.*
Consider a situation where $0 \le a_n \le b_n$,

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, and

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

The process works by comparing to known series which is known to either converge or diverge, so it important ensure that the one derived is either a p-series or harmonic series, since the convergence and divergence of such series are known.

Now let us remind ourselves of the skills in arranging fractions in descending order.
Consider $\frac{1}{2}, \frac{1}{3}$ *and* $\frac{1}{4}$.
How will you arrange them in order of magnitude?
I hope you can use at least two methods in doing that.
Let us write them under a single denominator:

Thus, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ become $\frac{6}{12}, \frac{4}{12,}, \frac{3}{12}$.

Hence, $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$.
Now replace 3 with the variable n, then

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} = \frac{\mathbf{1}}{\boldsymbol{n-1}} > \frac{\mathbf{1}}{\boldsymbol{n}} > \frac{\mathbf{1}}{\boldsymbol{n+1}}.$$

Take note of this. These ideas and many others are very necessary in this session.

Example 1: Is the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ convergent?

Solution: Note: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

Now $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a $\mathrm{p}-$series with $\mathrm{p}>$ which is convergent.

So by the comparison test $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ *is convergent*

Example 2: Show by the comparison test that the series

$\frac{1}{(1)(3)}+\frac{2}{(3)(5)}+\frac{3}{(5)(7)}+\frac{4}{(7)(9)}+\cdots$ diverges

Solution: From the question, the following sequences can be deduced:

1, 2, 3, 4, ..., n (for numerator)
1, 3, 5, 7. . . 2n-1 (for first values in denominator); and
3, 5, 7, 9, ..., 2n +1 (for second values in denominator)

Thus the series $\frac{1}{(1)(3)}+\frac{2}{(3)(5)}+\frac{3}{(5)(7)}+\frac{4}{(7)(9)}+\cdots$ can be written as

$$\frac{1}{(1)(3)}+\frac{2}{(3)(5)}+\frac{3}{(5)(7)}+\frac{4}{(7)(9)}+\cdots+\frac{n}{(2n-1)(2n+1)}$$
$$=\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}>\sum_{n=1}^{\infty}\frac{n}{(2n)(3n)}$$

But $\sum_{n=1}^{\infty}\frac{n}{(2n)(3n)}=\sum_{n=1}^{\infty}\frac{1}{6n}=\frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n}$

Since $\frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, then by the comparison test, the series $\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}$ diverges.

Note that, if $\lim_{n\to\infty}\frac{\sum a_n}{\sum b_n}=c>0$ (and c is finite), then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

NOTE: This works if a_n and b_n are always positive.

Key ideas

- The convergence or divergence of a given series may be determined by *comparing* its terms with terms of another series that is known to be convergent or divergent. Such a test is called *comparison test.*

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently guide students to use the comparison test to determine the convergence or divergence of a given series?

Discussions

1. Examine the series $\frac{5}{1.2.4}+\frac{7}{2.3.5}+\frac{9}{3.4.6}+\ldots..$ for convergence or divergence.
2. Is the series $\sum_{n=1}^{\infty}\frac{1}{3^n+2}$ *convergent*?

SESSION 4: ROOT TEST

In this session, we shall be addressing the issue of the root test.

Learning outcomes
By the end of the session, the participant will be able to:
a) state the conditions for determining convergence or divergence or otherwise of a given series using the root test;
b) determine convergence or divergence or otherwise of a given series using the root test.

Now read on …

Root Test

Suppose that $U_{n\,,} = L$ for each value of *n* of the series

$\sum_{n=1}^{\infty} U_{n\,,}$ and $\lim_{n\to\infty} |U_n| = L$,then

(i) $\sum_{n=1}^{\infty} U_n$ converges if L <1

(ii) $\sum_{n=1}^{\infty} U_n$ diverges if L> 1

(iii) No conclusion can be drawn if L = 1.

These are the conditions for determining convergence, divergence or otherwise of a series.

Example 1: Investigate the convergence or divergence of the following series using the root test.

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+.......+\left(\frac{n}{2n+1}\right)^n+....=\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$$

Solution: Using the root test:

$$U_n = \left(\frac{n}{2n+1}\right)^n \Rightarrow \sqrt[n]{U_n} = \frac{n}{2n+1}$$

Therefore, $\lim_{n\to\infty}\sqrt[n]{U_n} = \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \frac{1}{2+\frac{1}{n}} = \frac{1}{2+\left(\frac{1}{\infty}\right)} = \frac{1}{2} < 1$

Hence the series $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+..........\,...+\left(\frac{n}{2n+1}\right)^n+.....=\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$ is convergent.

Example 2: Determine the convergence or divergence of the series $\sum_{n=0}^{\infty}\frac{3^{2n}}{n^n}$.

Solution: Using the root test, $\lim_{n\to\infty} |(\frac{3^{2n}}{n^n})|^{\frac{1}{n}} = (\frac{3^2}{n})^{n.\frac{1}{n}} = \frac{3^2}{n}$

$$\lim_{n\to\infty}\frac{3^2}{n}=\frac{9}{\infty}=0<1, \text{hence the series } \sum_{n=0}^{\infty}\frac{3^{2n}}{n^n} \text{ converges}$$

Example 3: Is the series $\sum_{\mathbf{n=1}}^{\infty}\frac{\mathbf{2^n}}{\mathbf{2^{1+3n}}}$ convergent or divergent?.

Solution: Using the root test,

$$\lim_{n\to\infty}\left|\left(\frac{\mathbf{2^n}}{\mathbf{2^{1+3n}}}\right)\right|^{\frac{1}{n}}=\left(\frac{2}{\mathbf{2^{\frac{1}{n}}}+3}\right)^{n.\frac{1}{n}}=\frac{2}{\mathbf{2^{\frac{1}{n}}}.2^3}=\left|\frac{2}{2^3}\right|=\frac{1}{4}<1,$$

hence the series converges

Example 4: Determine the convergence or divergence of the series:

$$\sum_{n=4}^{\infty}\frac{(-5)^{1+2n}}{2^{5n-3}}$$

Solution: Using the root test,

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-4)^{1+2n}}{3^{4n-3}}\right|}=\sqrt[n]{\left|\frac{(-4)^1\,(-4)^{2n}}{3^{4n}.\ 3^{-2}}\right|}=\left|\frac{(-4)^{\frac{1}{n}}(-4)^2}{3^4.3^{\frac{-2}{n}}}\right|=\left|\frac{(-4)^{\frac{1}{\infty}}(-4)^2}{3^4.3^{\frac{-2}{\infty}}}\right|=\left|\frac{1(-4)^2}{3^4.1}\right|=\frac{16}{81}$$

$\text{Since } \frac{16}{81}<1, \text{the series converges.}$

Key ideas

- The root test is useful for determining convergence or divergence or otherwise of a given series.

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently guide students to use the Root test to determine the convergence or divergence of a given series?

Discussions

1. In each of the following, determine whether each series is convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{5^{2n+1}}}$ (b) $\sum_{n=1}^{\infty}\frac{n^3}{2^n}$

2. Determine if the ff converges or diverges:

a. $\sum_{n=1}^{\infty}\left(\frac{3n+1}{5-3n}\right)^{2n}$ b. $\sum_{\mathbf{n=0}}^{\infty}\frac{\mathbf{n^{2-n}}}{\mathbf{7^{3n}}}$

SESSION 5: RATIO TEST

In this session, we will discuss a few issues regarding factorials and then continue to deal in detail with ratio test.

Learning outcomes

By the end of the session, the participant will be able to:

a) state the conditions for determining convergence or divergence or otherwise of a given series using the ratio test;
b) determine convergence or divergence or otherwise of a given series using the ratio test.

Now read on …

Factorial

We begin the session, by turning our attention to the concept of factorial and the skills needed to deal with factorials. There may be questions involving factorials as in n! or in the power, like $4^{n!}$. The *factorial* symbol (!) tells you to multiply like this:
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.
Notice how things cancel when you have factorials in the numerator and denominator of a fraction:

$$\frac{6!}{5!} = \frac{6.5.4.3.2.1}{5.4.3.2.1} = 6$$
$$\frac{5!}{6!} = \frac{5.4.3.2.1}{6.5.4.3.2.1} = \frac{1}{6}$$

In both cases, everything cancels but the 6.
Thus, $\frac{(n+1)!}{n!} = n + 1$.
Also,

$$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}$$

In the same way everything cancels but the (n + 1). Lastly, it seems weird, but 0! = 1. Do a little search to get to understand why this is so.

Ratio test

Suppose we have the series $\sum a_n$. Define $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

NOTE: In the case of $L = 1$ the ratio test is pretty much worthless and we would need to resort to a different test to determine the convergence or divergence of the series.
Also, the absolute value bars in the definition of L are absolutely required. If they are not there it will be impossible for us to get the correct answer.

Example 1: Does the series $\sum_{n=0}^{\infty}\frac{2^{3n}}{n!}$ converge or diverge?

Solution: To do this, you look at the limit of the ratio of the (n + 1)th term to the nth term:

Let $U_n = \frac{2^{3n}}{n!}$ from this we derive $U_{n+1} = \frac{2^{3(n+1)}}{(n+1)!}$

Now, $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{3(n+1)}}{(n+1)!}}{\frac{2^{3n}}{n!}}\right| = \left|\frac{2^{3(n+1)}n!}{2^{3n}(n+1)!}\right|$

$$= \left|\frac{2^{3n}2^{3}n!}{2^{3n}(n+1)n!}\right| = \left|\frac{2^3}{(n+1)}\right| = \left|\frac{8}{(\infty+1)}\right| = \left|\frac{8}{\infty}\right| = 0$$

Since the $0 < 1$, the series $\sum_{n=0}^{\infty}\frac{2^{3n}}{n!}$ converges

Example 2: Does the series $\sum_{n=0}^{\infty}\frac{n^n}{n!}$ converge?

Solution: Let $U_n = \frac{n^n}{n!}$ and $U_{n+1} = \frac{(n+1)^{(n+1)}}{(n+1)!}$

Now $L = \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \left|\frac{\frac{(n+1)^n\,(n+1)}{(n+1)!}}{\frac{n^n}{n!}}\right|$

$$= \left|\frac{\frac{(n+1)^n\,(n+1)}{(n+1)n!}}{\frac{n^n}{n!}}\right| = \left|\frac{(n+1)^n\,(n+1)n!}{n^n(n+1)n!}\right| = \left|\frac{(n+1)^n}{n^n}\right| = \left|\left(\frac{(n+1)}{n}\right)^n\right| = \left|\left(1+\frac{1}{n}\right)^n\right|$$

Now, $\lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n\right| = e = 2.718 > 1,$ the series diverges

Key ideas

- The Ratio test is one of the effective ways of determining the convergence or divergence of a given series.
- Suppose we have the series $\sum a_n$. Defined, $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ Then,

  if $L < 1$ the series is absolutely convergent (and hence convergent).

  if $L > 1$ the series is divergent.

  if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently guide students to use the Ratio test to determine the convergence or divergence of a given series?

Discussions

1. Determine if the series converges or diverges:

a) $\sum_{n=0}^{\infty}\frac{(-8)^n}{5^{2(n+1)}(n+1)}$

b) $\sum_{n=0}^{\infty}\frac{2n!}{3^n}$

UNIT 6: PARTIAL FRACTIONS AND MATHEMATICAL INDUCTION

This unit introduces participants to how systems of equations can be used to decompose rational expressions into sums of simpler expressions. The last part of the unit will focus on Peano's postulates and its applications to the principle of Mathematical Induction.

Learning outcome(s)
By the end of the unit, the participant will be able to:

1. express a proper rational expression with linear factors in the denominator as partial fractions;
2. express a proper rational expression with repeated factors in the denominator as partial fractions;
3. express a proper rational expression with quadratic factors in the denominator as partial fractions;
4. express an improper rational expression with linear factors in the denominator as partial fractions;
5. identify and explain Peano's axioms and;
6. use the principle of mathematical induction to prove general formulae involving sequence of numbers.

SESSION 1: LINEAR FACTORS

In this session, we shall focus on the partial fraction decomposition of $\frac{f(x)}{g(x)}$, where $g(x) \neq 0$ and the denominator contains linear factors.

Learning outcomes
By the end of the session, the participant will be able to find the partial fraction decomposition in which all the denominators are linear factors.

Now read on ….

Consider the following combination of algebraic fractions:

$$\begin{aligned}\frac{2}{x-3}-\frac{4}{x-1} &= \frac{2(x-1)-4(x-3)}{(x-3)(x-1)}\\ &= \frac{2x-2-4x+12}{(x-3)(x-1)}\\ &= \frac{10-2x}{(x-3)(x-1)}\\ &= \frac{10-2x}{x^2-4x+3}\end{aligned}$$

The fractions on the left are called the *partial fractions* of the fraction on the right.

The reverse process of moving from $\frac{10-2x}{x^2-4x+3}$ to $\frac{2}{x-3}-\frac{4}{x-1}$ is called resolving into **partial fractions**.

In order to resolve an algebraic expression into partial fractions:

(i) the denominator of the fraction must factorize (for example, x^2-4x+3 factorizes as $(x-3)(x-1)$).

(ii) the numerator must be **at least** one degree less than the denominator (in the above example, $10-2x$ is of degree 1 since the highest powered x term is 1 and x^2-4x+3 is of degree 2).

When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator. We will be looking at three types of partial fractions namely; partial fractions with linear factors, partial fractions with repeated factors and partial fractions with quadratic factors.

In this case, the denominator *Q(x)* can be factored into linear factors, such that, all of them are distinct or different. The decomposition of *Q(x)* is as follows;

$$Q(x)=(x+a_i)(x+a_i)...(x+a_n)$$

Note that no two a_i 's are equal, where $i=1,2,...,n$.

Then $\frac{P(x)}{Q(x)}=\frac{A_1}{x+a_1}+\frac{A_2}{x+a_2}+...+\frac{A_n}{x+a_n}$, where $A_1,A_2,...,A_n$ are constants.

Examples 1: Express $\frac{2x+5}{x^2-x-2}$ in partial fractions.

Solution: First, the denominator is factorized to give:

$$\frac{2x+5}{x^2-x-2}=\frac{2x+5}{(x-2)(x+1)}$$

Resolve the fraction to partial fraction

$$\frac{2x+5}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1} \quad \Rightarrow 2x+5=A(x+1)+B(x-2)$$

$$x=-1 \quad \Rightarrow 2(-1)+5=B(-1-2)$$

Then, $3=-3\mathbf{B}$ and $B=-1$

Now, if $\Rightarrow$ $2(2)+5=A(2+1)$ $\Rightarrow$ $9=3A$ and $A=3$

Hence, $\frac{2x+5}{x^2-x-2}=\frac{3}{x-2}-\frac{1}{x+1}$

NB: Other methods can be used to find *A* and *B* such as the “cover up method” and the “comparison of coefficients”.

Examples 2: If $y=\frac{2x-3}{(x^2-1)(x+2)}$, express y in partial fractions.

Solution: $\frac{2x-3}{(x^2-1)(x+2)}=\frac{2x-3}{(x-1)(x+1)(x+2)}$

$$\frac{2x-3}{(x^2-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+2)}$$

$$\Rightarrow 2x-3 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

Solving using x = -1, x = 1, and x = -2 respectively gives,

$$B = \frac{5}{2},\ A = \frac{-1}{6} \text{ and } C = \frac{-7}{3}$$

Hence,

$$y = \frac{-1}{6(x-1)} + \frac{5}{2(x+1)} - \frac{7}{3(x+2)}$$

Key ideas

- To resolve an algebraic expression into partial fractions, the denominator of the fraction must factorize, and the numerator must be **at least** one degree less than the denominator.
- When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator.

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently teach partial fractions with linear factors?

Discussions

Find the partial fraction decomposition of: a) $\frac{4x^2+13x-9}{x^3+2x^2-3x}$ b) $\frac{5x+6}{(x+4)(x+6)}$ c) $\frac{x+7}{x^2-x-6}$ d) $\frac{3x}{(x+2)(x-1)}$

SESSION 2: REPEATED FACTORS

In this session, we shall continue with partial fraction decomposition but focusing on those whose denominators have repeated factors.

Learning outcomes

By the end of the session, the participant will be able to find the partial fraction decomposition in which the denominator has repeated linear factors.

Now read on …

Here *Q(x)* which is the denominator can be factored into repeated linear factors, that is,

$$Q(x) = (x+a_n)^{r1}(x+a_n)^{r2}...(x+a_n)^{rn}.$$

The following examples will help elaborate how this principle works.

i. $$\frac{2x+3}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

ii. $$\frac{5x^2+20x+6}{x(x+2)^2}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

Example 1: Express $\frac{2x+3}{(x-2)^2}$ in partial fractions.

Solution: $\frac{2x+3}{(x-2)^2} = \frac{A}{x-2}+\frac{B}{(x-2)^2} = \frac{A(x-2)+B}{(x-2)^2}$

Comparing numerators gives $2x+3 = A(x-2)+B$

If we let $x = 2$ and solve, we get $B = 7$

Now comparing coefficients of x and solving we get A =2

Hence, $\frac{2x+3}{(x-2)^2} = \frac{2}{x-2}+\frac{7}{(x-2)^2}$

Example 2: Resolve $\frac{5x^2-2x-19}{(x+3)(x-1)^2}$ into partial fractions.

Solution: $\frac{5x^2-2x-19}{(x+3)(x-1)^2} = \frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2} = \frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$

Equating numerators gives $5x^2-2x-19 = A(x-1)^2+B(x+3)(x-1)+C(x+3)$

If we let $x = -3$ and solve, we get $A = 2$

If we let $x = 1$ and solve, we get $C = -4$

Now comparing coefficients of x^2 and solving we get B =3

Thus, $\frac{5x^2-2x-19}{(x+3)(x-1)^2} = \frac{2}{x+3}+\frac{3}{x-1}-\frac{4}{(x-1)^2}$

Key ideas

- To resolve an algebraic expression into partial fractions, the denominator of the fraction must factorize, and the numerator must be **at least** one degree less than the denominator.
- When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator.

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently teach partial fractions with repeated linear factors?

Discussions

Find the partial fraction decomposition of: a) $\frac{-x^2+2x+4}{x^3-4x^2+4x}$ b) $\frac{x^2+10x-36}{x(x-3)^2}$ c) $\frac{6x-11}{(x-1)^2}$

SESSION 3: QUADRATIC FACTORS

In this session, we shall learn how to decompose rational functions where the denominator is a quadratic factor that cannot be factorised.

Learning outcome

By the end of the session, the participant will be able to find the partial fraction decomposition in which the denominator contains a quadratic which cannot be factorised.

Now read on ….

The denominator is a quadratic factor which does not factorize without introducing imaginary surd terms. Hence Q(x) which is the denominator can only be factored into quadratic expression. That is, $Q(x)=(x^2+b_1x+c_1)+(x^2+b_2x+c_2)+...(x^2+b_nx+c_n)$.

The following examples will help elaborate how this principle works.

i. $$\frac{2x+1}{x^2+2x+5}=\frac{Ax+B}{x^2+2x+5}$$

ii. $$\frac{3x^2+3x-7}{(x^2+5)(x^2+3x+20)}=\frac{Ax+B}{x^2+5}+\frac{Cx+D}{x^2+3x+20}$$

Example 1: Express $\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)}$ in partial fractions.

Solution: $\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+3}=\frac{A(x^2+2x+3)+(Bx+C)(x+1)}{(x+1)(x^2+2x+3)}$

Equating numerators gives $5x^2+7x+8=A(x^2+2x+3)+(Bx+C)(x+1)$

That is, $5x^2+7x+8=A(x^2+2x+3)+(Bx+C)(x+1)$

If we let $x=-1$ and solve, we get $A=3$

If we let $x=0$ and solve, we get $C=-1$

If we let $x=1$ and solve, we get $B=2$

Thus, $\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)} = \frac{3}{x+1}+\frac{2x-1}{x^2+2x+3}$

NB: In the case of repeated quadratic factors, combine the methods used in repeated linear factors and quadratic factors to resolve into partial fractions.

For example; $\frac{2x+3}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$

Recall that in order to resolve an algebraic expression into partial fractions, the numerator must be at least one degree less than the denominator. However, when the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator.

Example 2: Express $\frac{x^2+3x-10}{x^2-2x-3}$ in partial fractions.

Solution: $\frac{x^2+3x-10}{x^2-2x-3}=\frac{x^2-2x-3+5x-7}{x^2-2x-3}=1+\frac{5x-7}{x^2-2x-3}=1+\frac{5x-7}{(x+1)(x-3)}$

But $\frac{5x-7}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$
This implies $5x - 7 = A(x-3) + B(x+1)$
Solving gives $A = 3 \; and \; B = 2$
Thus $\frac{x^2+3x-10}{x^2-2x-3} = 1 + \frac{3}{x+1} + \frac{2}{x-3}$

Key ideas

- To resolve an algebraic expression into partial fractions, the denominator of the fraction must factorize, and the numerator must be **at least** one degree less than the denominator.
- When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator.
- Introduce imaginary surd terms to factorize quadratic factors which cannot be factorized.

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently teach students to find partial fraction decomposition in which the denominator contains a quadratic which cannot be factorised?
-

Discussions

Express the following into partial fractions.

1. $\dfrac{2x^2-4x+3}{(x-2)(x+1)}$
2. $\dfrac{5}{(x+1)(x-2)}$
3. $\dfrac{2x^2+3x+3}{(x+3)(x+2)x}$
4. $\dfrac{2x}{(x^3-8)}$
5. $\dfrac{x^3+1}{(x-1)^2}$

SESSION 4: IMPROPER FRACTIONS

In this session, we shall consider examples of partial fractions involving improper rational expressions.

Learning outcome

By the end of the session, the participant will be able to find the partial fraction decomposition involving improper fractions.

Now read on ….

Example 1: Express $\frac{x^3 -2x^2+4x+3}{(x-2)(x^2-4)}$ as partial fractions.

Solution: Expanding the denominator, we have $(x-2)(x^2-4) = x^3 - 2x^2 - 4x + 8$

We can see that the denominator is a cubic expression.

Therefore, the expression $\frac{x^3-2x+4x+3}{(x-2)(x^2-4)}$ is an improper fraction, since the degree of the numerator is 3 and that of the denominator is 3.

Using the long division, we have $\frac{x^3-2x^3+4x+3}{(x-2)(x^2-4)} = 1 + \frac{8x-1}{(x-2)(x^2-4)}$

We now express $\frac{8x-1}{(x-2)(x^2-4)}$ as a partial fraction of the form

Thus, $\frac{8x-1}{(x-2)(x^2-4)} = \frac{A}{(x-2)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$

Solving, we have

$$A = -\frac{17}{16}, B = \frac{17}{16}, C = \frac{15}{4}$$

Therefore,

$$\frac{x^3 - 2x^2 + 4x + 3}{(x-2)(x^2-4)} = 1 + \frac{17}{16(x+2)} + \frac{17}{16(x-2)} + \frac{15}{4(x-2)^2}$$

Key ideas

- The expression $\frac{x^3-2x+4x+3}{(x-2)(x^2-4)}$ is an improper fraction, since the degree of the numerator is 3 and the denominator is 3.
- Use long division approach to simplify partial fractions involving improper fractions into partial fraction form.

Reflections

- What are some of the experiences of handling some of the examples and questions in this session? How have these experiences prepared me to competently teach students to find partial fraction decomposition involving improper fractions.

Discussions

Express each of the following as partial fractions.

1. $\frac{3x^2+x+9}{(x+3)(x^2+x+5)}$ 2. $\frac{x^4-3x^3-3}{x^2(x-1)}$ 3. $\frac{5x^2-6x-21}{(x-4)^2(2x-3)}$

SESSION 5: PEANO'S POSTULATES

In this session, we will focus on Peano's postulates also known as Peano's axioms.

Learning outcome

By the end of the session, the participant will be able to explain Peano's postulates.

Now read on ….

Axiom: A statement whose truth is either to be taken as self-evident or to be assumed. There are some properties of *natural numbers*, N which are classified as ***Peano Axioms***.

Axiom 1: The set of natural numbers N is not empty. It contains a particular element called one and denoted by 1. Symbolically, $1 \in N$.

Axiom 2: For each $a \in N$, there exists a unique $a^* \in N$, called the successor of a. That is $a^* = a + 1 \in N$.

Axiom 3: The number 1 is not a successor of any number in N. In other words, the set of natural numbers begins with the number 1.

Axiom 4: Each element of N is a successor of at most one element in N. This means that if $a^* = b^*$, then $a = b$.

Axiom 5: Let M be the set of natural numbers with the following properties
(i) 1 is in M
(ii) If k is in M then k^* is also in M
Then $M = N$.

According to the Encyclopedia Britannica, 15th edition, the five Peano's postulates are:

1. 0 is a number.
2. The successor of any number is also a number.
3. No two distinct numbers have the same successor.
4. 0 is not the successor of any number.
5. If any property is possessed by 0 and also by the successor of any number having that property, then all numbers have that property.

The fifth axiom is known as the *Principle of Mathematical Induction* because it can be used to establish properties for an infinite number of cases without having to give an infinite number of proofs. In particular, given that P is a property and zero has P and that whenever a natural number has P its successor also has P, it follows that all natural numbers have P. The 5^{th} axiom may be stated in the following manner:
A statement involving the natural number n is true for every $n \in N$ *provided that*:

I. the statement is true in the special case $n = 1$
II. the truth of the statement for $n = k, k \in N$
$\Rightarrow$ the truth of the statement for $n = k + 1$.

In practice, the use of the *principle of mathematical induction* falls into **two** steps:
Step 1: Verify that the statement to be proved is true for $n = 1$ (or $n = 2$).
Step 2: (a) Assume that the statement to be proved is true for k, $k \in N$.
(b) Prove that the statement is true for $n = k + 1$.

Key ideas

- ***Axiom*** is a statement whose truth is either to be taken as self-evident or to be assumed.
- According to the Encyclopedia Britannica, 15th edition, there are five Peano's postulates.
- The fifth axiom is known as the *Principle of Mathematical Induction* because it can be used to establish properties for an **infinite** number of cases without having to give an infinite number of proofs.

Reflections

- How has the content of the session broadened my understanding on the importance and usage of Peano's postulates/axioms?

Discussions

1. State the first Peano's axiom.
2. State Peano's fifth axiom.
3. State the principle of mathematical induction and the steps that has to be followed.

SESSION 6: MATHEMATICAL INDUCTION

In Session 5, you learned about Peano's axioms/postulates and how these axioms were foundational in generating the set of natural numbers. In this session, we shall learn how to apply the principle of mathematical induction to prove some formulae.

Learning outcome

By the end of the session, the participant will be able to use the principle of mathematical induction to prove general formulae involving sequence of numbers.

Now read on …

Mathematical induction is the method of proof frequently used to prove general formulae, such as a formula for the sum of a sequence of *n* numbers.
This method consists of three major steps.

1. Verify that the proposed formula is true for an initial (small) value of *n*. That is to show that it is true for $n=1$
2. While assuming that the proposed formula is true for a specific value of *n*, prove that the formula is also true for the next value of *n*. That is, show that if $n = k$ is true then $n = k+1$ is also true.
3. Conclude that (because of mathematical induction) the formula in fact does hold for all values of *n*.

Example 1: Prove by mathematical induction that $1+2+3+4+...+n=\frac{n(n+1)}{2}$,

for all natural numbers.

Solution: Let S(*n*) be the statement

The statement is true for S(1) (i.e. $n=1$), since $1=\frac{1(1+1)}{2}=1$

Assume S(*k*) is also true (i.e. $n = k$)

Thus, $1+2+3+4+...+k=\frac{k(k+1)}{2}$

If S(k) is true, the S(k+1) is also true (i.e. If the statement is true for n = k, then it is true for n = k+1)

That is, $1+2+3+4+\ldots+k+(k+1)=\frac{(k+1)[(k+1+1)]}{2}$

But $1+2+3+4+\ldots+k+(k+1)=\frac{(k)(k+1)}{2}+(k+1)$

This implies $1+2+3+4+\ldots+k+(k+1)=\frac{(k)(k+1)+2(k+1)}{2}$

$$1+2+3+4+\ldots+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

Hence for all natural numbers $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$.

Example 2: Prove by mathematical induction that $1^2+2^2+3^2+4^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$, for all natural numbers.

Solution: Let p(n) be the statement

The statement is true for p(1), since $1^2=\frac{1}{6}(1)(2)(3) \quad \Rightarrow 1=1$

Assume p(k) is also true (i.e. n = k)

Thus, $1^2+2^2+3^2+4^2+\ldots+k^2=\frac{1}{6}k(k+1)(2k+1)$

If p(k) is true, the p(k+1) is also true

Hence, $1^2+2^2+3^2+4^2+\ldots+k^2+(k+1)^2=\frac{1}{6}k(k+1)(2k+1)$

Implying that $1^2+2^2+3^2+4^2+\ldots+k^2+(k+1)^2=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$

$$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$=\frac{(k+1)(2k^2+7k+6)}{6}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=\frac{1}{6}(k+1)(k+2)(2k+3)$$

$$=\frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]$$

Hence for all natural numbers $\sum_{r=1}^{n} r^2=\frac{1}{6}n(n+1)(2n+1)$.

Example 3: Prove by mathematical induction that $1+3+5+\ldots+(2n-1)=n^2$, for all natural numbers.

Solution:

Let p(n) be the statement

The statement is true for p(1) (i.e. n =1), since $2(1)-1=1^2 \quad \Rightarrow 1=1$

Assume p(k) is also true (i.e. n = k)

Thus, $1+3+5+\ldots+(2k-1)=k^2$

If p(k) is true, the p(k+1) is also true

Hence, $1+3+5+\ldots+(2k-1)+[2(k+1)-1]=k^2$

$$\text{Implying that, } 1+3+5+...+(2k-1)+[2(k+1)-1] = k^2+[2(k+1)-1]$$
$$= k^2+2k+1$$
$$= (k+1)^2$$

Hence the formula is true for all positive integral values of *n* by induction.

NB: The above formula can be stated as "prove by mathematical induction that the sum of the first *n* odd numbers is equal to the *n*th square number".

Key ideas

- Mathematical induction is the method of proof frequently used to prove general formulae, such as a formula for the sum of a sequence of *n* numbers.

Reflections

- How has the content of the session broadened my understanding on the importance and usage of Peano's postulates/axioms?

Discussions

1. Prove by mathematical induction that $\frac{1}{1.2}+\frac{1}{2.3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for all integers $n \geq 1$
2. Use mathematical induction to prove that $S_n = 2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$ for every positive integer n.

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