Module for B.Ed Early Childhood Education Programme

EBS101SW: ELEMENTARY ALGEBRA

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UNIT 1: BINARY OPERATIONS

This unit engages users and practitioners in activities that explore the closure, commutative, associative, and distributive properties of binary operations on real numbers. Finally, the unit uses cooperative learning groups to engage students to identify the identity elements and inverses for defined binary operations.

Learning outcome(s)

By the end of the unit, the participant will be able to determine whether the following properties hold for given sets of elements under defined binary operations:

1. Commutative property
2. Associative property
3. Distributive property
4. Closure property
5. Identity element and inverses

SESSION 1: PROPERTIES OF BINARY OPERATIONS

In this session, we shall discuss the commutative, associative, distributive, and closure properties of binary operations defined over a set of numbers. We shall first attempt to define these properties and follow it with the proof of whether a given operation defined over a given set is commutative, associative, distributive over another operation, and finally closed.

Learning outcomes

By the end of the session, the participant will be able to determine whether the following properties hold for given sets of elements under defined binary operations:

1. Commutative property
2. Associative property
3. Distributive property
4. Closure property

Commutative Property

A binary operation $\boxtimes$ defined on a set S is said to be commutative if and only if $m \boxtimes n = n \boxtimes m$. For example, we know that the operations $\oplus$ and $\otimes$ (addition and multiplication respectively) are commutative on the set of real numbers. This means $m \oplus n = n \oplus m$ and $m \otimes n = n \otimes m$ .Test with some examples on your own.

Let us now use examples to determine whether the commutative property holds for given sets of elements under a defined binary operation.

Example 1. The operation $a * b$ is defined on the set of real numbers., R, by $a * b = a + b^2$. Show whether the operation $*$ is commutative.

Solution

For the $*$ to be commutative, $a * b = b * a$.

For left hand side (LHS), $a * b = a + b^2$ by definition.

Also, for right hand side (RHS), $b * a = b + a^2$

Comparing LHS to RHS, $a+b^2 \neq b+a^2$

Hence $a*b \neq b*a$

Therefore, the operation is not commutative over the given definition.

Example 2. Show whether the operation $\boxtimes$ is commutative under the set of real numbers if $m \boxtimes n = m+n-2mn$.

Solution

For the operation $\boxtimes$ to be commutative, $m \boxtimes n = n \boxtimes m$.

For left hand side (LHS), $m \boxtimes n = m+n-2mn$ by definition.

Also, for right hand side (RHS), $n \boxtimes m = n+m-2nm$

But $n+m = m+n$ and $nm = mn$ (commutative properties of addition and multiplication).

The RHS can thus be written as $n \boxtimes m = m+n-2mn$

Comparing LHS to RHS, $m \boxtimes n = n \boxtimes m$. Hence the operation $\boxtimes$ is commutative.

Associative Property

A binary operation $\boxtimes$ is defined on a set S is said to be associative if and only if $m \boxtimes (n \boxtimes p) = (m \boxtimes n) \boxtimes p$. For example the operation + is associative on the set of real numbers because $3+(4+5)=(3+4)+5$. True or False? Also, the operation $\times$ is associative over the set of real numbers since $3\times(4\times5)=(3\times4)\times5$.

Let us now use examples to determine whether or not the associative property holds for given sets of elements under a defined binary operation.

Example 1

The operation $*$ is defined on the set of real numbers. R. by $a*b = a+2b$. Find:

a. $5*(3*7)$

b. $(5*3)*7$

Is the operation $*$ associative?

Solution

a.
$$\begin{aligned} 5*(3*7) &= 5*(3+2\times7) \\ &= 5*(17) \\ &= 5+2(17) \\ &= 5+34 \\ &= 39 \end{aligned}$$

b.
$$\begin{aligned} (5*3)*7 &= (5+2\times3)*7 \\ &= (11)*7 \\ &= (11)+2\times7 \\ &= (11)+14 \\ &= 26 \end{aligned}$$

The operation $*$ is not associative since $5*(3*7) \neq (5*3)*7$

Example 2

If $m\Delta n = mn + m + n$, show that the operation Δ is associative on the set of real numbers.

Solution

Let m, n, and p belong to the set of real numbers then for associativity, $m\Delta(n\Delta p) = (m\Delta n)\Delta p$.

For LHS, $m\Delta(n\Delta p) = m\Delta(np + n + p)$

$$= m(np + n + p) + m + (np + n + p)$$
$$= mnp + mn + mp + m + np + n + p$$
$$= m + n + p + mn + mp + np + mnp$$

For RHS, $(m\Delta n)\Delta p = (m + n + mn)\Delta p$

$$= (m + n + mn) + p + (m + n + mn)p$$
$$= (m + n + mn) + p + mp + np + mnp$$
$$= m + n + p + mn + mp + np + mnp$$

Comparing LHS with RHS, they are the same. This means $m\Delta(n\Delta p) = (m\Delta n)\Delta p$. Thus, the operation Δ is associative on the set of real number for the given definition.

Distributive property

This property involves the use of two different operations. It is the same as expansion. The binary operation $*$ is distributive over the operation $\square$ on a given set, if and only if $x * (y \square z) = (x * y) \square (x * z)$. For example, you already know that $4 \times (3 + 5) = (4 \times 3) + (4 \times 5)$. That is the operation $\times$ is distributive over the operation $+$. Is operation $\times$ distributive over the operation $-$? Why?

We shall work some examples to determine whether a given operation is distributive over another given operation on a given set.

Example 1

Two binary operations $*$ and $\square$ are defined on the set of real numbers as $a * b = a + b + 2$ and $m \square n = 2mn$ respectively. Find:

a. $3 * (5 \square 8)$

b. $(3 * 5) \square (3 * 8)$

Is the operation $*$ distributive over $\square$ on the set of real numbers? Why?

Solution

a. By definitions, $3 * (5 \square 8) = 3 * (2 \times 5 \times 8)$

$$= 3 * (80)$$
$$= 3 + 80 + 2$$
$$= 85$$

b. $(3 * 5) \square (3 * 8) = (3 + 5 + 2) \square (3 + 8 + 2)$

$$= 10 \square 13$$
$$= 2 \times 10 \times 13$$
$$= 260$$

Since the answers in (a) and (b) are not the same the operation $*$ is not distributive over $\square$ on the set of real numbers. $3 * (5 \square 8) \neq (3 * 5) \square (3 * 8)$

Example 2

Show whether or not the operation Δ is distributive over the operation $\square$ on the set of real numbers if $a\Delta b = a + 2b$ and $m \square n = 2m + n$.

Solution

For the operation Δ to be distributive over the $\square$, $a\Delta(b \square c) = (a\Delta b) \square (a\Delta c)$

LHS, $a\Delta(b \square c) = a\Delta(2b + c)$

$$= a + 2(2b + c)$$

$$= a + 4b + 2c$$

Also, from the RHS, $(a\Delta b) \square (a\Delta c) = (a + 2b) \square (a + 2c)$

$$= 2(a + 2b) + (a + 2c)$$

$$= 2a + 4b + a + 2c$$

$$= 3a + 4b + 2c$$

Since, $LHS \neq RHS$, the operation Δ is not distributive over the operation $\square$ on the set of real.

Closure Property

The closure property can be likened to the natural principle that animals give birth to their kinds. The property is that when a binary operation $*$ is defined over a set S, the set of S is said to be closed under the operation $*$ if $a * b$ belongs to S for all a, b belonging to S. For example, the set of integers is closed under the operation + (addition), because the sum of any two integers is an integer. Also, the set of natural numbers is not closed under the operation – (subtraction) because some natural when subtracted give a difference of a negative number, which is not a natural number.

Example 1.

Show whether or not the set of natural numbers is closed under the operation $*$ given that $a * b = a + b + ab.$

Solution

Using for example 2 and 3 where 2,3 belong to the set of natural numbers, $2 * 3 = 2 + 3 + 2 \times 3$

But $2 * 3 = 2 + 3 + 6 = 11$, where 11 belongs to the set of natural numbers. It is obvious that this work for any pair of natural number chosen. We can thus conclude that given $a * b = a + b + ab$, the set of natural numbers is closed under the operation $*$.

For clarity, in the next example we shall consider a finite set of natural numbers.

Example 2.

Show whether or not the set of natural numbers, $N = \{1, 2, 3, ...10\}$ is closed under the operation $*$ given that $a * b = a + b + ab.$

Solution

This may work for some numbers and for others it may not work.

For example, using the numbers 2 and 5 which belong to N, $2 * 5 = 2 + 5 + 2 \times 5.$ Thus, $2 * 5 = 17.$

Although a natural is not a member of N. We can now conclude that the set, $N=\{1,2,3,...10\}$ is not closed under the operation $*$ defined by $a*b=a+b+ab.$

(**Note that one counter-example is sufficient to show that a set is not closed under particular operation**).

Key ideas

- For the operation $*$ to be commutative, $a*b=b*a$.
- An operation $\square$ is defined on a set S is said to be associative if and only if $m \square (n \square p)=(m \square n) \square p.$
- An operation $*$ is distributive over the operation $\square$ on a given set, if and only if $x*(y \square z)=(x*y) \square (x*z)$
- The set of S is said to be closed under the operation $*$ if $a*b$ belongs to S for all a,b belonging to S.

Reflection

1. If $a \square b=ab+a+b$, find the value of
   (i) $3 \square 5$
   (ii) $5 \square 3$
   (iii) $(3 \square 5) \square 7$
   (iv) $3 \square (5 \square 7)$

   two conclusions from your answers.

2. The operation $\square$ is defined as $x \square y=x^2+y^2-2xy$.

a. Evaluate $\sqrt{3} \square \sqrt{5}$

b. Show that the $\square$ is both commutative and associative on the set of real numbers.

3. The operations $*$ and Δ are defined on the set of natural numbers $N=\{1,2,3,...20\}$ by $a*b=a+b+2$ and $m\Delta n=m+n-1$.

a. Evaluate $3*(4\Delta 2)$

b. Find $(3*4)\Delta(3*2)$

c. Is the operation $*$ distributive over the operation Δ ?

d. Is N closed under the operation $*$? Give a reason to your answer.

e. Is N closed under the operation Δ ? Give a reason to your answer.

4. Copy and complete the table for the operation $\square$ on the set $P=\{1,2,3,4,5\}$ if $m \square n=m+n-1$.

$\square$	1	2	3	4	5
1					
2		3			
3				6	
4					

5	5				9

Use the results in your table to find whether or not:

a. The set P is closed under the given operation $\otimes$.

b. The operation $\otimes$ commutative over the given definition.

SESSION 2: IDENTITY ELEMENTS AND INVERSES

The number zero (0) is called additive identity of real numbers because the sum of any real number and zero or the sum of zero and any real number is the number. 2 has it inverse as -2 because the sum of 2 and it inverse, -2, is 0, the identity element. Similarly, -5 has it additive inverse as +5 since $-5+5=0$ (0 being the additive identity). The number 1 is called the multiplicative identity of real numbers. Why? Because the product of any real number and 1 or the product of 1 and any real number is the number. What is the multiplicative inverse of for example 5? To answer this question, you need to think of a number which will give a product of 1 (the multiplicative identity) when multiplied by 5. Of course, the number is $\frac{1}{5}$ (the reciprocal of 5). These introductory ideas will prepare the ground for you to define the identity element (also called the neutral element) and the inverse element of a given binary operation.

Learning outcomes

By the end of the session, the participant will be able to:

1. determine the identity element for a binary operation defined over a set S.
2. find the inverse element for a binary operation defined over a set S.

Identity Element

For a binary operation $*$ defined on a set S, there is a unique element 'e' called the identity element, such that $a*e=e*a=a$, where *a* and *e* belong to S.
We shall now learn how to determine the identity element using examples.

Example 1.

The binary operation $\otimes$ is defined on R, the set of real numbers by $x \otimes y = x+y-2$ for all $x, y \in R$. Find the identity element.

Solution

a. Let *e* be the identity element.

Then by definition of identity element, $x \otimes e = x$ …………. (1)

But $x \otimes e = x+e-2$

$\therefore x+e-2=x$ from …… (i)

Solving for *e*, $e-2=0$

$e=2$

The identity element is thus 2.

Example 2

The binary operation ∇ is defined on R, the set of real numbers by $x \nabla y = x + y - 2xy$ for all $x, y \in R$
Find the identity element of 7.

Solution

Let '*e'* be the identity element of 7, then $7 \nabla e = 7$ (by definition).

But $7 \nabla e = 7 + e - 2 \times 7 \times e$

$= 7 - 13e$

This means $7 - 13e = 7$

$-13e = 0$

And therefore $e = 0$

Inverse Element

For a binary operation $*$ defined on a set S, there is an element '*i*' called the inverse element, such that $a * i = i * a = e$, where *e* is the identity element, and $a, e \in S$. What this definition means is that without the knowledge of the identity element, one cannot determine the inverse element. Do you agree?

Let us work an example on the inverse element.

Example 1

The binary operation $\boxtimes$ is defined on R, the set of real numbers by $x \boxtimes y = x + y - 2xy$ for all $x, y \in R$

Find:

a. the identity element.
b. the inverse element.

Solution

a. Let *e* be the identity element, then by definition $x \boxtimes e = x$
But $x \boxtimes e = x + e - 2xe$
This means $x + e - 2xe = x$
And therefore, $e = 0$. Explain.

b. Let *I* be the inverse element, then by definition $x \boxtimes i = e$
This means $x \boxtimes i = 0$ since $e = 0$.
But $x \boxtimes i = x + i - 2xi = 0$
Solving for *i*, we obtain

$x + i - 2xi = 0$

$i - 2xi = -x$

$$i(1-2x) = -x$$

$$i = \frac{-x}{1-2x}$$

Therefore, the inverse, *i*, is $\frac{-x}{1-2x} x \neq \frac{1}{2}$,

Key ideas

- For a binary operation $*$ defined on a set S, there is a unique element 'e' called the identity element, such that $a*e = e*a = a$, where *a* and *e* belong to S.
- For a binary operation $*$ defined on a set S, there is an element '*i*' called the inverse element, such that $a*i = i*a = e$, where *e* is the identity element, and $a, e \in S$.

Reflection

The binary operation ⊗ is defined on R, the set of real numbers by $x ⊗ y = x + y + 3$ for all $x, y \in R$
Find the:
a. identity element.
b. Inverse element

1. The binary operation ⊗ is defined on the set of real numbers R by $p ⊗ q = 2pq$. Find the:
a. identity element of 5.
b. Inverse element of 5.

2. Determine the:
a. inverse element
b. inverse element of 3

he binary operation ⊗ is defined on the set of real numbers R by $m ⊗ n = \frac{1}{m} + \frac{1}{n}$.

UNIT 2: APPLICATION OF SETS

This unit deals with sets. Although the idea of a set features prominently in our daily activities, it is also a mathematical concept. We can refer to a 'group of students' as 'a set of students' in set terminology. In the same vein, 'class', population', and so on, are all examples of sets as each one expresses the basic idea of a collection of objects or people with a common defining property. The unit is designed to strengthen your knowledge of set theory and enable you to step in the classroom with confidence, realizing that you do not have to be a genius to understand the language of sets.

Learning outcome(s)

By the end of the Unit, you should be able to use real life situations to determine

1. a set and describe it
2. subsets of given sets
3. complements of sets
4. perform some basic operations on sets
5. solve two and three set problems.

SESSION 1: DEFINITION AND TYPES OF SETS

In this session, you will be guided to define a set and you will also learn of the various types of sets.

Learning outcome(s)

By the end of this session, you should be able to:

1. define a set
2. identify the various types of sets.

Definition of a set

A **set** is intuitively understood to be a well-defined collection of objects. In other words, a set may be considered as a group or class of objects or persons with the following properties:

i. the objects or persons in the set are distinct,
ii. either a given object or person belongs to the set or not.

Thus, members of a set have some common characteristics. For example, given the set E={even whole numbers}, $4 \in E$, but $3 \notin E$.

Ways of Defining (Describing) a Set

A set may be defined in various ways:

a. **Roster Method (Listing):** In this method, the objects or members or elements are listed as shown in the following examples:

A={1, 2, 3, 4, 5};

B={2, 4, 6, 8,…., 20}

C={2, 3, 5, 7, 11, 13,…}

b. **Defining Property**: Sometimes a set is briefly described by highlighting the common property of the elements of the set. For example:

A={the first five natural numbers}

B={the first twenty positive even numbers}
C={prime numbers}

c. **Set-builder notation form:** Elements of a set may also be described in the form
A={x : 1≤x≤5, x ∈N}
B={x : x=2n, n=1, 2, 3,…10}

d. **Recursive or Inductive rule:** Elements in a set may be defined by recursive or inductive rule. For example, A={a1, a2, a3,…}, where a1=1, and an=an-1+2 for n=2, 3, 4,…

Note: Sets are denoted by upper case (capital) letters.

Types of Sets

We shall now learn about some types of sets and how sets are generally named based on certain characteristics.

Empty Set
Although, generally, sets contain elements or members, there is an exceptional set which has no element. This is called an empty or null set and it is denoted by { } or ∅.

Unit Set or Singleton
This refers to a set which contains exactly one element.
Example of unit sets are: {a}, {0}.

Finite Set
A set whose elements are countable and the process of listing its elements terminates is referred to as a finite set. An example is A={a, e, i, o, u}, with the number of elements in the set, A, being five (5), written as n(A)=5.

Infinite Set
A set whose elements are not countable or whose elements cannot be fully listed is referred to as an infinite set. An example is W={0, 1, 2, 3,…}, that is, the set of all whole numbers is an infinite set.

Universal Set
This is the mother of all sets. In other words, a universal set, U, is a set which contains all elements, including itself. For example, U={1, 2, 3,…, 10} is a universal set with the sets A={2, 4, 6} and B={1, 2, 3, 4, 5} as its subsets.

Equal Sets
The Sets P and Q are said to be equal if every element of P is an element of Q and vice versa. For example given Set P={1, 3, 5} and Set Q={1, 3, 5}, these two sets are equal because they contain the same elements and also have the same number of elements.

Now, determine whether the following sets are equal or not and give a reason for your answer:

a. F={3, 5, 7, 9}
b. G={5, 7, 9, 11}

We hope that you got the answer to be **not** equal, and the reason is that although sets F and G have the same number of elements, they both contain an element not found in the other set.

If S1 and S2 are sets, and S1 S2 and S2 S1, then S1 and S2 are equal and we write S1 = S2, otherwise S1 S2.

Equivalent Sets

The sets C and D are said to be equivalent if and only if they have the same number of elements. In this case, unlike the equal sets, the two sets should not necessarily have the same elements. For example, C={1, 2, 3, 4, 5, 6} and D={a, b, c, d, e, f} are equivalent. Note that the number of elements in a set refers to its cardinality. Therefore, if two sets have the same cardinality, then they are equivalent.

Key ideas

- A **set** is intuitively understood to be a well-defined collection of objects.
- A set may be defined in various ways:
  - Roster Method (Listing)
  - Defining Property
  - Set-builder notation form
  - Recursive or Inductive rule

- Some types of Sets may be said to be:
  - Empty Set
  - Unit Set or Singleton
  - Finite Set
  - Infinite Set
  - Universal Set
  - Equal Sets
  - Equivalent Sets

Reflection

1. List three ways of describing a set. Give an example in each case.

2. Give an example each of a set defined by using

a. roster method

b. set-builder notation

3. List the first five members of

an = 2an-1 – 1, a1 = 2, n = 2, 3, 4,…

4. All equal sets are equivalent, but not **all** equivalent sets are equal sets. True or False? Explain your answer.

5. Explain the difference between a unit set and an empty set. Give an example in each case.

6. Define the following types of sets and give an example in each case:

a) unit set
b) an infinite set.

SESSION 2: SUBSETS

At this stage, you should be able to explain what a set is and also be able to describe sets. You should also distinguish between finite and infinite sets, as well as identify universal, unit and null sets.

This session introduces you to the concept of subsets and the use of Venn diagrams to represent sets.

Learning outcomes

By the end of the session, the participant will be able to:
By the end of the session, you should be able to:
1. state and explain a proper subset of a set;
2. state and explain an improper subset of a set.

Now read on …

Subset of a set

Consider the sets $F = \{p, q, r\}$ and $G = \{p, q, r, s, t\}$. Compare them and write down all your observations. Did you observe that every element of *F* is an element of *G*? If *F* and *G* are two subsets such that every element of *F* is an element of *G*, then the set *F* is called a subset of G. In other words, we can say that the set *F* is contained in the set G, or the set G contains the set *F*. In mathematical symbols, "the set *F* is a subset of G" is written as $F \subseteq G$, which may be read as "*F* is contained in *G*". Note that $F \subseteq G$ may also be written as $G \supseteq F$ which is read as "*G* contains *F*" or "*F* is contained in *G*", or "*G* is a superset of *F*". If at least an element of *F* is not an element of *G*, then *F* is not a subset of *G*. In this case we write $F \not\subseteq G$ or $G \not\supseteq F$. For example, the set $H = \{p, q, w\}$ is not a subset of $G = \{p, q, r, s, t\}$ because $w \notin G$

Suppose you are given that $K = \{1, 2, 3, 4\}$, write down as many subsets of *K* you can. Does your list of subsets of *K* include $\{1\}$, $\{2\}$, $\{1, 2\}$, $\{2, 4\}$, $\{1, 4\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$ and so on?

Note that every element of the set $\{1, 2, 3, 4\}$ is also an element of the set $K = \{1, 2, 3, 4\}$, and so the set $\{1, 2, 3, 4\}$ which is *K* itself is a subset of *K*.

In general, every set is a subset of itself. Given any set *A*, the subset *A*, which is the set itself and the null set, $\{\ \}\, or\, \varnothing$, are called the improper (or trivial) subsets of *A*. All other possible subsets are called proper subsets of A. If A is a proper subset of B, we write $A \subset B$ or $B \supset A$.

Number of subsets of a set

We can list the subsets of sets as shown below:

Set	Number of elements	Subsets	Number of subsets
$\{\ \}$	0	$\{\ \}$	$1=2^0$
$\{1\}$	1	$\{\ \}$, $\{1\}$	$2=2^1$
$\{1,2\}$	2	$\{\ \}$, $\{1\}$, $\{2\}$, $\{1,2\}$	$4=2^4$
$\{1,2,3\}$	3	$\{\ \}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$	$8=2^3$
$\{1,2,3,4\}$	4	$\{\ \}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,3,4\}$	$16=2^4$

Establishing a relationship between the number of elements (n) and the number of subsets, S, you will find that if a set has n elements in it, then the number of subsets (S) it has is given by $S=2^n$.

Key ideas

- If *F* and *G* are two subsets such that every element of *F* is an element of *G*, then the set *F* is called a subset of G.
- The list of subset of a certain set *A* which includes the subset *A*, which is the set itself and the null set, $\{\ \}\, or\, \varnothing$, are called the improper (or trivial) subsets of *A*.
  - All other possible subsets are called proper subsets of A.
- If A is a proper subset of B, we write $A\subset B$ or $B\supset A$.
- The number of subsets (S) is given by $S=2^n$.

Reflection

1. If $U=\{1,2,3,....,20\}$, find:

a. The subset *A* consisting of all even numbers in *U*.

b. The subset *B* consisting of all elements that are multiples of four.

c. What relationship is there between sets *A* and *B*.

2. Indicate each of the following as true or false;

a. $\{5,7,9,11,...\}\subseteq\{1,3,5,....\}$

b. $\varnothing\subseteq\{\ \}$

c. $\{1,2\}\subseteq\{1,2,\{1\},\{2\},\{1,2\}\}$

3. Suppose we are given that $A\subseteq B$,

a. Is it possible that $A\subseteq\varnothing$?

b. Is it possible that $B\subseteq\varnothing$?

4. Suppose $E\subseteq F$. If $F=\varnothing$, what can we conclude about *E*?

5. If $A \not\subset B,$ can we then conclude that $B \not\subset A$? Justify your answer.

SESSION 3: OPERATIONS ON SETS

Welcome to this session which focusses on operations on sets. Just as we can perform operations on numbers and on various terms in algebra, we can also perform some operations on set. In this session, we will learn how to perform some basic operations on set.

Learning outcomes

By the end of the session, you should be able to:

a. perform the operations union and intersection of sets;
b. use Venn diagrams to represent simple statements;
c. state and verify the properties of union and intersection of sets.

Now read…

Union of sets

Given two sets A and B, we can form new sets by performing various operations on them. One of such operations involves collecting together all elements in *A* and in *B*. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8, 10\}$, then by collecting all elements in *A* and *B*, we obtain A union B written, $A \cup B$, as $A \cup B = \{1, 2, 3, 4, 6, 8, 10\}$. Note that, though each of the elements 2 and 4 belongs to both *A* and *B*, each is written once in the union set, $A \cup B$.

In general, the union of any sets A and B is the set which consists of all elements in either A or B, or both. Another way to express the meaning of the union of two sets, for instance the sets *A* and *B* is $A \cup B = \{x : x \in A \, or \, x \in B\}$.

Do an activity

List the members of the union of the following pairs of sets:

1. $\{m, e, a, n\}$ and $\{m, o, d, e\}$
2. $\{\text{prime numbers less than } 20\}$ and $\{\text{odd numbers less than } 20\}$
3. $\{2, 4, 9, 16, 25, 36\}$ and $\{2, 4, 8, 16, 32\}$
4. $\{\text{odd numbers }\}$ and $\{\text{even numbers }\}$

The idea of union of sets discussed so far can be extended to cover more than two sets. For example, given the sets $X_1, X_2, X_3, \ldots, X_n$, we can form union by collecting together all the elements in $X_1, X_2, X_3, \ldots, X_n$. We write the union as $X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_n$.

We can use Venn diagrams to represent the set $A \cup B$ as in Fig. 1.1

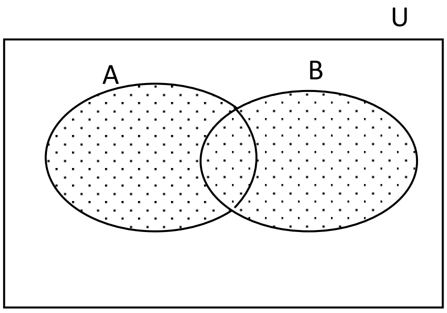


Fig. 1.1: $A \cup B$ (Shaded region)

Note that the overlap of the two circles in the diagram indicates that ***A*** and ***B*** have some elements in common. If ***A*** and ***B*** have no elements in common, then $A \cup B$ would be represented by Venn diagram in Fig. 1.2.

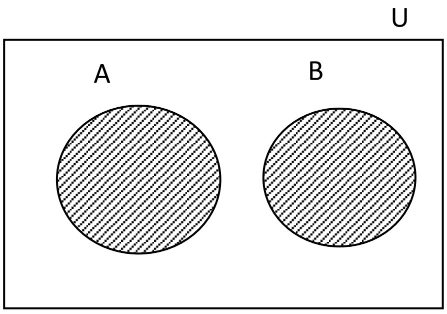


Fig. 1.2. $A \cup B$ **(Shaded region)**

If A is a subset of B, then $A \cup B$ would be represented by Venn diagram in Fig.1.3.

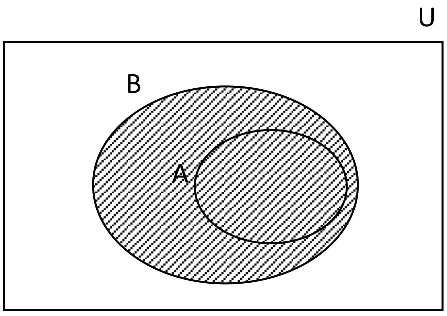


Fig. 13: $A \cup B$ with $A \subseteq B$ **(Shaded region)**

Intersection of sets

Given two sets *A* and *B*, we can form another new set *D*, say, by collecting together all elements which belong to both A and B. For example, if $A = \{1,2,3,4\}$ and $B = \{2,4,6,8,10\}$, then by collecting all elements which belong to *A* as well as to *B*, we obtain a new set called *A* intersection *B* (written $A \cap B$) as $A \cup B = \{2,4\}$. Note that, the elements 2 and 4 belong to both *A* and *B*.

In general, the intersection of any sets, *A* and *B* is the set that consists of all elements that belong to both A and B. Another way to express the meaning of the intersection of two sets, for instance the sets *A* and *B* is $A \cup B = \{x : x \in A \, and \, x \in B\}$.

Do an activity

List the members of the intersection of the following sets:

1. $\{m, e, a, n\}$ and $\{m, o, d, e\}$
2. $\{\text{prime numbers less than 20}\}$ and $\{\text{odd numbers less than 20}\}$
3. $\{2, 4, 9, 16, 25, 36\}$ and $\{2, 4, 8, 16, 32\}$
4. $\{\text{odd numbers }\}$ and $\{\text{even numbers }\}$

We can also extend the idea of intersection of sets discussed so far can be extended to cover more than two sets. For example, given the sets X_1, X_2, X_3, ..., X_n, we can form intersection by collecting together all the elements that are common to all the sets. in X_1, X_2, X_3, ..., X_n. We write the intersection as $X_1 \cap X_2 \cap X_3 \cap ... \cap X_n$.We can use Venn diagrams to represent the set $A \cap B$ as in Fig. 1.4.

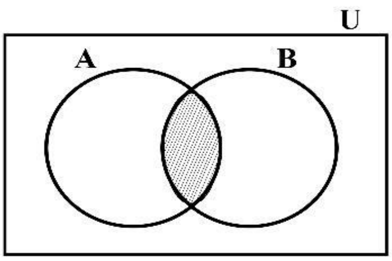


Fig. 1.4 $A \cap B$ (Shaded region)

If A is a subset of B, then $A \cap B$ would be represented by Fig. 1.5

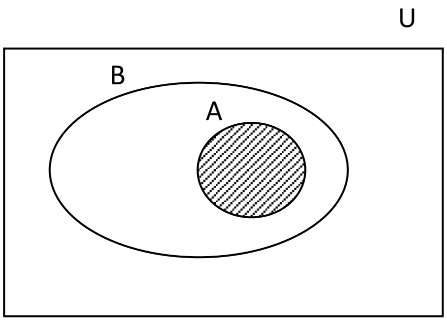


Fig. 1.5 $A \cap B$ with $A \subseteq B$ (Shaded region)

This example shows that if A is a subset of B, then $A \cap B$ is simply A; that is $A \cap B = A$.

Disjoint Sets

If $A \cup B = \varnothing$, we say that A and B have no elements in common, or simply, *A* and *B* are **disjoint sets**. Diagrammatically, they are represented by Fig. 1.6.

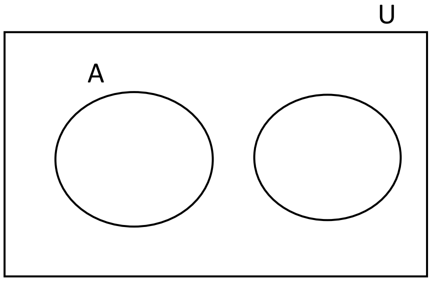


Fig. 1.6. $A \cup B = \varnothing$

Properties of union and intersection of sets

Union and intersection of sets, like real numbers, also have properties such as commutative, associative, and distributive. We will verify each using sets. These are summarized below.

Commutative property of union of sets

For any two sets, *A* and *B*,

$$A \cup B = B \cup A$$

Verification:

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8, 10\}$

$A \cup B = \{1, 2, 3, 4\} \cup \{2, 4, 6, 8, 10\}$

$= \{1, 2, 3, 4, 6, 8, 10\}$

Also

$B \cup A = \{2, 4, 6, 8, 10\} \cup \{1, 2, 3, 4\}$

$= \{1, 2, 3, 4, 6, 8, 10\}$

$\therefore A \cup B = B \cup A$

Commutative property of intersection of sets

For any two sets, *A* and *B*,

$$A \cap B = B \cap A$$

Verification:

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8, 10\}$

$A \cap B = \{1, 2, 3, 4\} \cap \{2, 4, 6, 8, 10\}$

$= \{2, 4\}$

Also

$B \cap A = \{2, 4, 6, 8, 10\} \cap \{1, 2, 3, 4\}$

$= \{ 2, 4 \}$

$\therefore A \cap B = B \cap A$

Associative property of union of sets

For any three sets, *A*, *B* and *C*,

$(A \cup B) \cup C = A \cup (B \cup C)$

Verification:

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 5, 7, 9\}$

Now, $A \cup B = \{1, 2, 3, 4, 6, 8, 10\}$

So $(A \cup B) \cup C = \{1, 2, 3, 4, 6, 8, 10\} \cup \{1, 3, 5, 7, 9\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ …………………. (1)

Also $B \cup C = \{2, 4, 6, 8, 10\} \cup \{1, 3, 5, 7, 9\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

And so, $A \cup (B \cup C) = \{1, 2, 3, 4\} \cup \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ …………………… (2)

From equations (1) and (2), we observe that:

$(A \cup B) \cup C = A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

This verifies the association property of union of sets.

Associative property of intersection of sets

For any three sets, *A*, *B* and *C*,

$(A \cap B) \cap C = A \cap (B \cap C)$

Verification

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 5, 7, 9\}$

Now, $A \cap B = \{2, 4\}$

So $(A \cap B) \cap C = \{2, 4\} \cap \{1, 3, 5, 7, 9\}$

$= \{ \ \}$ …………………. (1)

Also $B \cap C = \{2, 4, 6, 8, 10\} \cap \{1, 3, 5, 7, 9\}$

$= \{ \ \}$

And so, $A \cup (B \cup C) = \{1, 2, 3, 4\} \cap \{ \ \}$

$= \{ \ \}$ …………………… (2)

From equations (1) and (2), we observe that:

$(A \cap B) \cap C = A \cap (B \cap C) = \{ \ \}$

This verifies the association property of intersection of sets.

Distributive property of intersection over union of sets

For any three sets, *A*, *B* and *C*,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Verification:

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 5, 7, 9\}$

$B \cup C = \{2,4,6,8,10\} \cup \{1,3,5,7,9\}$

$= \{1,2,3,4,5,6,7,8,9,10\}$

Now, $A \cap (B \cup C) = \{1, 2, 3, 4\} \cap \{1,2,3,4,5,6,7,8,9,10\}$

$= \{1, 2, 3, 4\}$ …………………………(1)

Also

$A \cap B = \{1, 2, 3, 4\} \cap \{2,4,6,8,10\}$

$A \cap B = \{2, 4\}$

And $A \cap C = \{1, 2, 3, 4\} \cap \{1,3,5,7,9\}$

$A \cap C = \{1, 3\}$

So $(A \cap B) \cup (A \cap C) = \{2,4\} \cup \{1,3\}$

$= \{1, 2, 3, 4\}$ ………………….. (2)

Thus, from equations (1) and (2), we conclude that:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) = \{1, 2, 3, 4\}$$

This verifies the distributive property of intersection over union.

Distributive property of union over intersection of sets

For any three sets, *A*, *B* and *C*,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Verification

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 5, 7, 9\}$

$B \cap C = \{2,4,6,8,10\} \cap \{1,3,5,7,9\}$

$= \{\ \}$

$A \cup (B \cap C) = \{1,2,3,4\} \cup \{\ \}$

$A \cup (B \cap C) = \{1,2,3,4\}$ ………………………. (1)

$A \cup B = \{1, 2, 3, 4\} \cup \{2,4,6,8,10\}$

$= \{1, 2, 3, 4, 6,8,10\}$

$A \cup C = \{1, 2, 3, 4\} \cup \{1,3,5,7,9\}$

$= \{1, 2, 3, 4, 5, 7, 9\}$

$(A \cup B) \cap (A \cup C) = \{1,2,3,4,6,8,10\} \cap \{1,2,3,4,5,7,9\}$

$= \{1,2,3,4\}$ …………………… (2)

Thus, from equations (2) and (3), we have

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) = \{1,2,3,4\}$$

This verifies the distributive property of union over intersection.

Key ideas

- The union of any sets A and B is the set which consists of all elements in either A or B, or both. $A \cup B = \{x : x \in A \, or \, x \in B\}$.
- The intersection of any sets, *A* and *B* is the set that consists of all elements that belong to both A and B. $A \cup B = \{x : x \in A \, and \, x \in B\}$.
- If $A \cup B = \varnothing$, we say that A and B have no elements in common, or simply, *A* and *B* are **disjoint sets**
- Union and intersection of sets have properties such as commutative ($A \cup B = B \cup A$), associative [$(A \cup B) \cup C = A \cup (B \cup C)$], and distributive [($A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$].
- Intersection of sets have properties such as commutative ($A \cap B = B \cap A$), associative [$(A \cap B) \cap C = A \cap (B \cap C)$], and distributive [$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$]

Reflection

1. $U = \{1,2,3,...,12\}$ and $A = \{2,3,5,7,11\}$ **, verify that:**

a. $A \cup U = U$

b. $\varnothing \cup U = U$

c. $A \cap U = A$

d. $\varnothing \cap U = \varnothing$

2. Let $U = \{a,b,c,d,e,f,g,h,i\}$, $P = \{a,b,c,d,e,f\}$, $Q = \{c,d,e\}$, $R = \{d,e,f,g\}$ and $S = \{f,g,h,i\}$. Verify the following equality of sets.

(a) $P \cup (Q \cap S) = (P \cup Q) \cap (P \cup S)$

(b) $(Q \cup S) \cap (S \cup R) \cap (R \cup Q) = (Q \cap S) \cup (S \cap R) \cup (R \cap Q)$

SESSION 4: COMPLEMENT OF A SET

Dear student, we continue our discussion on set by considering another aspect. Sometimes we have a particular universal set in mind as well as a set selected from it, but we may want to speak also of set of the remaining elements. In the session, we will discuss this latter set and how we can identify its members.

Learning outcome(s)

By the end of the session, you should be able to:

1. identify the complement of a given set which is a subset of the universal set;
2. use Venn diagram to represent the complement of a set.

If we are given a well-defined universal set U and a set A with, then the set of members of U that are not in A is called the complement of A and it is written as $A^{/}$. We can also define $A^{/}$ in the set builder notation form as

$$A^{/} = \{x : x \in U \text{ and } x \notin A\}$$

Clearly, we can see that $A \cup A^{/} = U$. Also, there are no members in the set $A \cap A^{/}$, that is, $A \cap A^{/} = \varnothing$. The Venn diagram for the complement of a set is illustrated in Fig. 1.7.

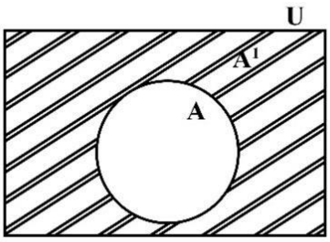


Fig. 1.7: $A^{/}$ (shaded region)

The complement of the set is illustrated in Fig. 1.8

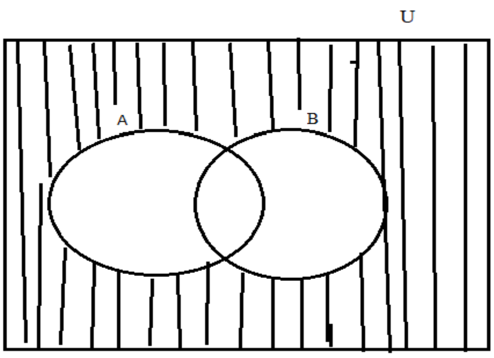


Fig. 1.8 The shaded region represents $A \cup B$ and the shaded represents $(A \cup B)^{/}$

Example 1.

If $U = \{1, 2, 3, 4, ...\}$ and $B = \{x : x \text{ is an even number}\}$, find $B^{/}$.

Solution

$U = \{1, 2, 3, 4, ...\}$

$B = \{2, 4, 6, 8, ...\}$

$\therefore B^{/} = \{1, 3, 5, 7, ...\}$, the set of odd numbers.

Example 2.

Given that $U = \{0, 1, 2, 3, ..., 12\}$, $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7, 9, 11\}$. List the members of each of the following sets.

(a) $A^{/} \cap B$ (b) $(A \cap B)^{/}$ (c) $A^{/} \cup B^{/}$

Solutions

(a) $A^{/} \cap B = \{0, 5, 6, 7, 8, 9, 10, 11, 12\} \cap \{1, 3, 5, 7, 9, 11\}$

$= \{5, 7, 9, 11\}$

(b) $(A \cap B)'$

First, we find $A \cap B = \{1, 3\}$

Next, we fine the complement of $A \cap B$

$\therefore (A \cap B)' = \{0, 2, 4, 5, 6, 7, 8, 9.10, 11, 12\}$

(c) $A' = \{0, 5, 6, 7, 8, 9, 10, 11, 12\}$

$B' = \{0, 2, 4, 6, 8, 10, 12\}$

$\therefore A' \cup B' = \{0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

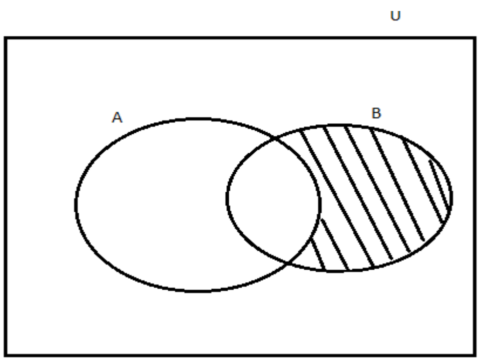
Example 3

Draw a Venn diagram to illustrate each of the following:

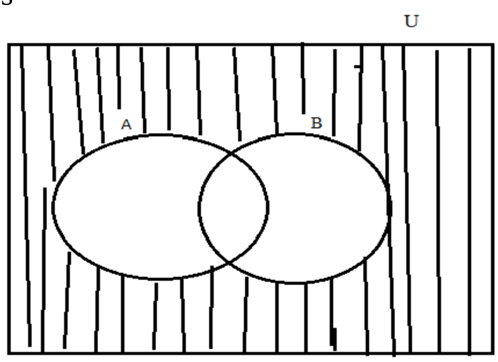
(a) $A' \cap B$ (b) $(A \cup B)'$ (c) $A' \cap B'$

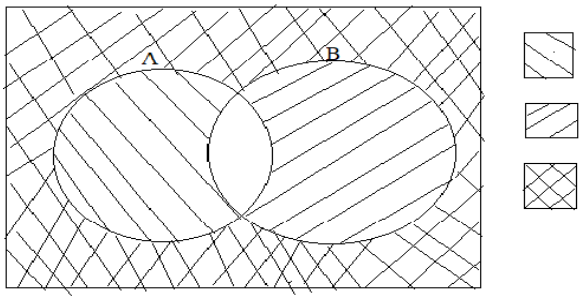
Solution

(a) $A' \cap B$



(b) Shaded region is $(A \cup B)'$



(c) 

Region with the double lines is $A' \cap B'$

Example 4

If $U = \{0,1,2,3,....,12\}$ $A = \{1,2,3,4\}$, $B = \{1,3,5,7,9,11\}$, $C = \{0,2,4,6,....,12\}$, and $D = \varnothing$, list the members of the following sets:

(a) $[(A \cup B) \cap C]'$

(b) $[(A \cup B) \cup C]' \cap D$

Solution

(a) $A \cup B = \{1,2,3,4\} \cup \{1,3,5,7,9,11\}$

$= \{1,2,3,4,5,7,9,11\}$

$[(A \cup B) \cap C] = \{1,2,3,4,5,7,9,11\} \cap \{0,2,4,6,8,10,12\}$

$= \{2,4\}$

$\therefore [(A \cup B) \cap C]' = \{0,1,3,5,6,7,8,9,10,11,12\}$

(b) $A \cup B = \{1,2,3,4\} \cup \{1,3,5,7,9,11\}$

$= \{1,2,3,4,5,7,9,11\}$

So $[(A \cup B) \cup C] = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$

$\therefore [(A \cup B) \cup C]' = \{\ \}$

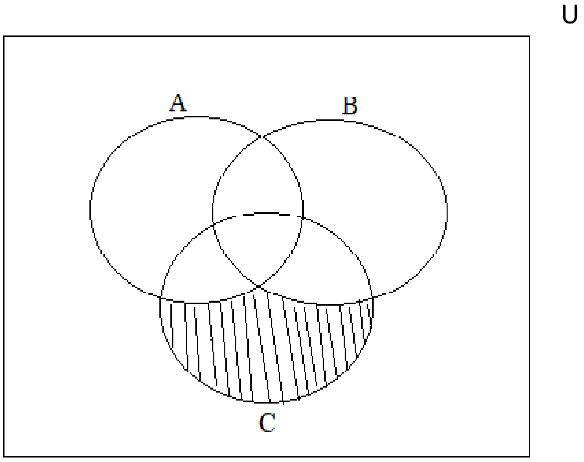
Finally, $[(A \cup B) \cup C]' \cap D = \{\ \} \cap \varnothing$

$= \{\ \}$ or $\varnothing$

Example 5.

Draw a Venn diagram to illustrate $(A \cup B)' \cap C$

Solution



Key ideas

- If we are given a well-defined universal set U and a set A with, then the set of members of U that are not in A is called the complement of A and it is written as $A^{/}$.
- The complement of *A* is given by $A^{/} = \{x : x \in U \; and \; x \notin A\}$
- $A \cup A^{/} = U$.
- $A \cap A^{/} = \varnothing$.

Reflection

$U = \{a,b,c,d,e,f,g,h,i\}$, $A = \{a,b,c,d,e,f\}$, $B = \{c,d,e,\}$, $C = \{d,e,f,g\}$, and $D = \{f,g,h,i\}$.

(a) Find the following sets:

i) $A \cup B^{/}$ ii) $B^{/} \cup C^{/}$ iii) $D^{/} \cup C^{/}$

(b) Verify the following equality of sets.

i) $B^{/} \cap C^{/} = B \cup F$

ii) $B^{/} \cup C^{/} = B \cap F$

2. Draw Venn diagram to illustrate each of the following:

i) $A \cap B^{/}$

ii) $A \cap B^{/} \cap C^{/}$

SESSION 5: SOLVING TWO-SET PROBLEMS

Welcome to session four of this Unit. So far, we have learned some concepts related to sets including some operations on sets. In this session, we will discuss how to use sets to solve everyday life problems. We will discuss various regions in the Venn diagram and how it is useful in providing a diagrammatic representation of word problems, making it easier to solve such problems.

Learning outcome(s)

By the end of this session, you should be able to:

1. draw a Venn diagram representing two intersecting sets and label it's regions;
2. represent given information on a Venn diagram;
3. solve simple two-set problems.

Let us begin our discussion by learning how to describe the various regions of a Venn diagram representing two intersecting sets. The diagram in Fig. 1.9 represents a universal set and two subsets, *A* and *B*, that have some elements in common.

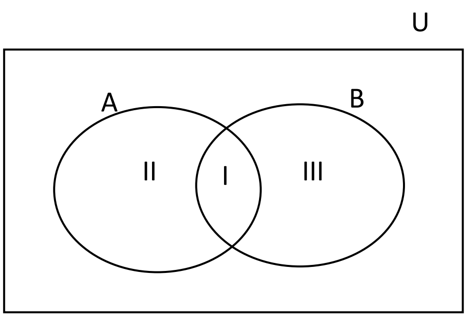


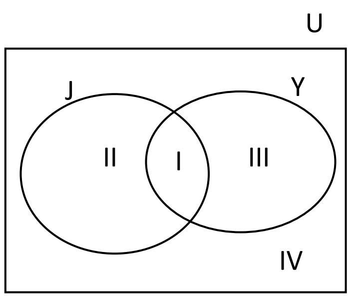
Fig. 1.9; Regions of two intersecting sets.

Observe from Fig. 1.9 that Region I is inside both *A* and *B*. It represents $A \cap B$, the intersection of *A* and *B*. Now look at Region II. All the elements in this region are inside *A* but outside *B*. It is $A \cap B'$, the intersection of set *A* and the complement of set *B*.

What can you say about the elements in Regions III and IV? Describe these regions in terms of A and B. Region III represents elements that are outside set *A* but inside set *B*. It is $A' \cap B$, the intersection of the complement of set A and set B. Region IV represents elements that are not in *A* and also not in *B*. It is $(A \cup B)'$ or $A' \cap B'$, the complement of the union of sets *A* and *B* or the complement of the set *A* intersection the complement of the set *B*.

Example 1

Consider the following example. In the figure shown:



$U = \{\text{names of the months of the year}\}$

$J = \{\text{months which begin with J}\}$

$U = \{\text{ months which end with y}\}$

1. Describe the elements of each of the regions I to IV.
2. List the elements of those regions.
3. Express each region in terms of J and Y.

Let us discuss these questions together.

1. $\text{I} = \{\text{names of months which begin with J and with Y}\}$

   Region II represents elements in J but not in Y, so

   $\text{II} = \{\text{names of months which begin with J but do not end with Y}\}$

   Region III represents elements in Y but not in J, so

   $\text{III} = \{\text{names of months which end with Y but do not begin with J}\}$

   Region IV represents elements that are not in J and not in Y but are U, so

   $\text{IV} = \{\text{names of months which do not begin with J nor end with Y}\}$

2. Now let us list the elements of each of the regions.

   $\text{I} = \{\text{January, July}\}$, $\text{II} = \{\text{June}\}$, $\text{III} = \{\text{February, May}\}$,

   $\text{IV} = \{\text{March, April, August, September, October, November, December}\}$

3. If we express each of the regions in terms of J and Y, we have:

   $\text{I} = \text{J} \cap \text{Y}$, $\text{II} = \text{J} \cap \text{Y}'$, $\text{III} = \text{J}' \cap \text{Y}$, $\text{IV} = \text{J}' \cap \text{Y}'$ or $(\text{J} \cup \text{Y})'$

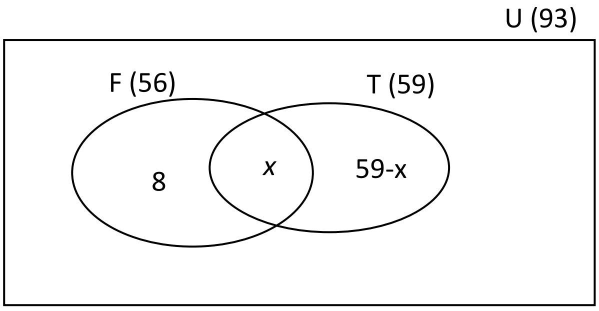
Example 2

Let us work through the following problems, which is an example of another group of problems that we can use Venn diagram to solve. In all 93 athletes reported to take part in a competition involving various events. Of this number, 56 registered to take part in field events and 59 registered to take part in track events. At the end of the competition, it was observed that 8 athletes took part only in field events. Find how many of the athletes:

a. took part in only track events.
b. took part in neither field nor track events.

To solve this problem, let us first extract or gather all information from the given statements, and then draw a Venn diagram to represent the information.

Let us represent the number of athletes who took part in field and track events *x*, (ie. $n(F \cap T) = x$).
We also have $n(U) = 93$, $n(F) = 56$, $n(T) = 59$ and $n(F \cap T') = 8$.
Then we can represent the above information in a Venn diagram as follows:



From the diagram and information provided, we obtain:

$$8 + x = 56$$
$$x = 56 - 8$$
$$x = 48$$

Therefore,

(a) the number of athletes who took part in only track events is $59 - 48 = 11$

Now, the total number of athletes who took part in at least one event is given by:

$$8 + 48 + 11 = 67$$

Therefore,

(b) the number of athletes who took part neither in field nor track events is:

$$93 - 67 = 26$$

Example 3

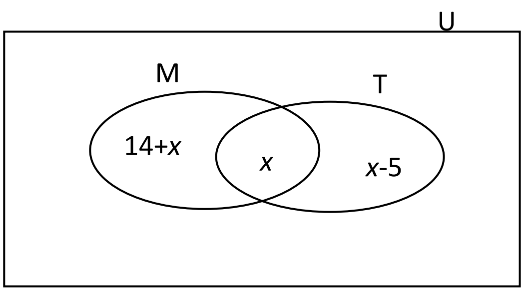
In a certain class, students are allowed to do either Mathematics or ICT or both. The number of students who do Mathematics only is 14 more than those doing both Mathematics and ICT. The number of students doing ICT only is 5 less than the number doing Mathematics and ICT. The number of students doing Mathematics is twice those doing ICT. Find the number of students who do both Mathematics and ICT.

Solution

Let $n(U)$ be the number of students in class, $n(M)$ be the number of students doing mathematics, and $n(T)$ the number of students doing ICT.

Then, $n(M \cap T) = x$

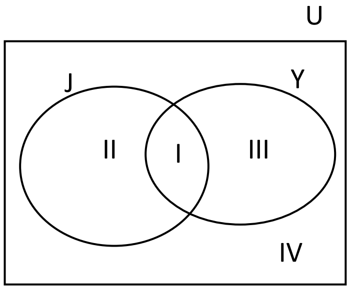
$$n(M \cap T') = 14 + x$$
$$n(M' \cap T) = x - 5$$
$$n(M) = 2n(T)$$ In each try and answer the question "Why?"



$$2x = 24$$
$$x = 12$$

Therefore, the number of students doing both Mathematics and ICT is 12.

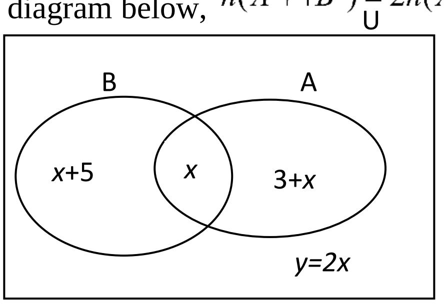
Key ideas



$\mathrm{I} = \mathrm{J} \cap \mathrm{Y}$, $\mathrm{II} = \mathrm{J} \cap \mathrm{Y}'$, $\mathrm{III} = \mathrm{J}' \cap \mathrm{Y}$, $\mathrm{IV} = \mathrm{J}' \cap \mathrm{Y}'$ or $(\mathrm{J} \cup \mathrm{Y})'$

Reflection

1. Draw a Venn diagram of a universal set, U and two Intersecting subsets, A and B. If $n(A \cap B) = 5$, $n(A' \cap B) = 11$, and $n(A \cap B') = 19$, find $n(A)$ and $n(B)$.
2. Draw a Venn diagram to illustrate the following information: $\mathrm{U} = \{x : 1 \le x \le 12, x \in N\}$, where N is the set of natural numbers, $A = \{1, 5, 7, 8\}$, $B = \{3, 5, 6, 8\}$, and $C = \{1, 2, 3, 8\}$.
3. In the Venn diagram below, $n(A' \cap B') = 2n(A \cap B)$ and $n(U) = 38$. Find the values of *x* and *y*.



SESSION 6: SOLVING THREE-SET PROBLEMS

We end our discussions on sets by looking at problems involving three sets. In this session we will discuss how we could solve three-set problems using Venn diagrams.

Learning outcomes

By the end of the session, the participant will be able to:

1. name the regions in Venn diagram representing three intersecting sets;
2. represent simple set problems involving three intersecting sets in Venn diagrams;

3. Solve simple three-set problems using Venn diagrams.

Three-set problems are problems involving three intersecting sets. Again, we begin our discussion by learning how to describe the various regions of a Venn diagram represented by three intersecting sets, *A*, *B* and *C*.

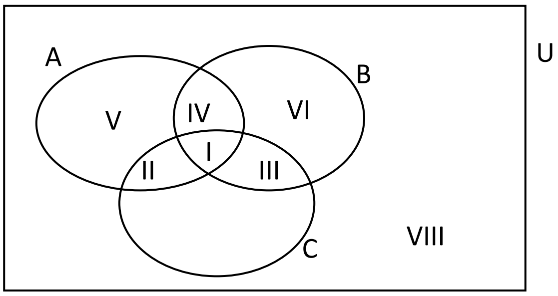


Fig. 1.10: Regions of three intersecting sets.

We observe from Fig. 1.10 that the Region I is in *A*, *B* and *C*. It is $A \cap B \cap C$, the intersection of *A*, *B* and *C*.

Now look at Region II. Do you observe that it is inside *A* and *C* But outside *B*? It is $A \cap B' \cap C$ the intersection of set *A*, the complement of set *B* and set *C*.

Similarly, Region III is inside B and C, but outside A, ie. $A' \cap B \cap C$.

Region IV is inside *A* and *B*, but outside *C*, ie. $A \cap B \cap C'$.

Now looking carefully at V, it is clear that it's inside *A* but outside *B* and *C*. We write $A \cap B' \cap C'$

Similarly, Region VI is inside B but outside *A* and *C*, ie. $A' \cap B \cap C'$.

Region VII is inside *C*, but outside A and B, ie. $A' \cap B' \cap C$.

It should be clear to you that though Region VIII is inside *U*, it is outside *A*, *B* and *C*. This is expressed as $A' \cap B' \cap C'$.

Example 1

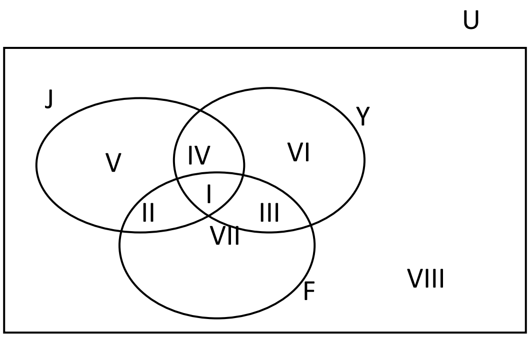
Consider the following example. In the figure below,

$U = \{\text{names of months of the year}\}$

$F = \{\text{names of months with more than five letters}\}$

$J = \{\text{names of months which begin with J}\}$

$Y = \{\text{names of months which end with y}\}$



1. Describe the elements of each of the regions I to VIII.
2. List the elements of each of these regions.
3. Express each region in terms of F, J, and Y.

Let us work through this problem together.

1. $I = \{\text{months with more than five letters and begin with J and end with y}\}$.

$II = \{\text{months with more than five letters and begin with J but do not end with y}\}$.

$III = \{\text{months with more than five letters and end with y but do not begin with J}\}$.

$IV = \{\text{months which begin with J and end with y, but do not have more than five letters}\}$

$V = \{\text{months which begin with J but do not end with y, and not with more than five letters}\}$.

$VI = \{\text{months which end with y, but do not begin with J nor have more than five letters}\}$.

$VII = \{\text{months with more than five letters but do not begin with J nor end with y}\}$.

$VIII = \{\text{months which do not begin with J, or end with y or with more than five letters}\}$.

2. $I = \{\text{January}\}$. $\quad V = \{\text{June}\}$.

$II = \{\ \}$. $\quad VI = \{\text{May}\}$.

$III = \{\text{February}\}$. $\quad VII = \{\text{August, September, October, November, December}\}$.

$IV = \{\text{July}\}$. $\quad VIII = \{\text{March, April}\}$.

3. $I = F \cap J \cap Y$ $\quad V = F' \cap J \cap Y'$

$II = F \cap J \cap Y'$ $\quad VI = F' \cap J' \cap Y$

$III = F \cap J' \cap Y$ $\quad VII = F \cap J' \cap Y'$

$IV = F' \cap J \cap Y$ $\quad VIII = F' \cap J' \cap Y'$

We now work through an example of another group of problems which we can apply Venn diagram to solve.

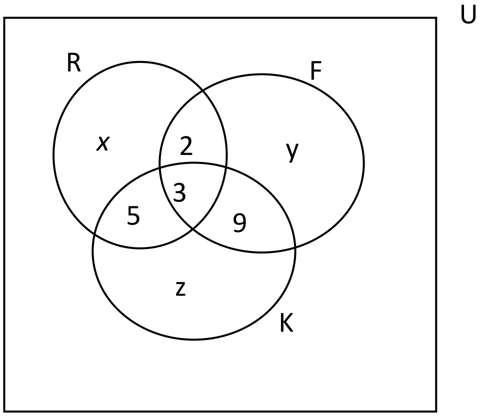
A number of school children were asked whether they liked rice, fufu or kenkey. Twelve said they liked rice, 16 liked fufu and 21 liked kenkey. Only 3 children said they liked all the three foods. Five children liked rice and fufu (this include those that liked all the three foods). Eight children liked rice and kenkey and 12 children liked kenkey and full. How many of the children liked:

a. rice only?
b. fufu only?
c. kenkey only?
d. rice or fufu or kenkey?

Solution

To solve this problem, let us first extract all necessary information and then draw a Venn diagram to represent them.

Let R represent those who liked rice, F represent those who liked fufu and K represent those who liked kenkey. Then $n(R)=12$, $n(F)=16$, $n(K)=21$, $n(R\cap F\cap K)=3$, $n(R\cap F)=5$, $n(R\cap K)=8$, $n(F\cap K)=12$. $\Rightarrow n(R\cap F\cap K')=2$, $n(R\cap F'\cap K)=5$, $n(R'\cap F\cap K)=9$



Let $n(R\cap F'\cap K')=x$, $n(R'\cap F\cap K')=y$ and $n(R'\cap F'\cap K)=z$

(a) $x+2+3+5=12$

$$x+10=12$$

$$x=2$$

$\therefore$ 2 children liked rice only.

(b) $y+2+3+9=16$

$$y+14=16$$

$$y=2$$

$\therefore$ 2 children liked fufu only.

(c) $z+5+3+9=21$

$$z+17=21$$

$$z=4$$

$\therefore$ 4 children liked kenkey only.

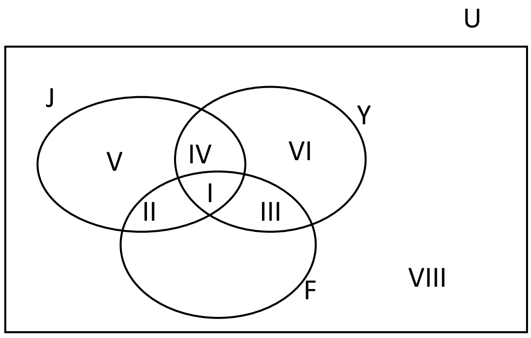
(d) $n(R\cap F\cap K)=x+2+3+5+9+y+z$

$$=2+2+3+5+9+2+4$$

$$=27$$

$\therefore$ 4 children liked rice or fufu or kenkey.

Key ideas



$I = F \cap J \cap Y$

$II = F \cap J \cap Y'$

$III = F \cap J' \cap Y$

$IV = F' \cap J \cap Y$

$V = F' \cap J \cap Y'$

$VI = F' \cap J' \cap Y$

$VII = F \cap J' \cap Y'$

$VIII = F' \cap J' \cap Y'$

Reflection

1. Draw a Venn diagram to illustrate the following information:
   $U = \{x : 1 \leq x \leq 12; x \in N\}$, where N is the set of natural numbers.
   $A = \{1, 5, 7, 8\}$, $B = \{3, 5, 6, 8\}$ and $C = \{1, 2, 3, 8\}$
2. A survey of 100 randomly selected students gave the following information:
   45 students take Mathematics, 41 students take English and 40 students take History. In addition, 15 students take Mathematics and English, 18 take Mathematics and History, and 17 take English and History. If 7 students take all three subjects,
   (a) how many take only Mathematics?
   (b) how many students take only English?
   (c) how many students take only History?
   (d) How many students do not take any of the three subjects?

3. In recent survey of 100 women, it is gathered that 59 use shampoo A, 51 use shampoo B, 35 use shampoo C; 24 use shampoo A and B, 19 use A and C and 13 use B and C. If 11 use all three, represent the information on a Venn diagram showing the number of women in each of the eight categories (regions).

UNIT 3: RATIO, PROPORTIONS, PERCENTAGES, AND RATES

The concepts of ratio, rate, proportion and percent are very useful to every one of us in our daily activities. Professionals of all kind make use of these concepts daily; the health assistants need the concept of ratio when they have to direct patients on drug usage/medication. What amount of water is needed for a mixture with a drug to be sure of the potency of the drug? Blood pressure measures are often stated in fractional/rate form 120/90. Checking on the number of children and number of adults admitted to the hospital in a given month and comparing these numbers in the form for every one adult admitted there are five children; More men report to the hospital with urinary tract infection than female – for every one female there are as many as six men; Forty percent of patients at the OPD on daily basis reported of malaria in June 2019.

This unit discusses the concepts of ratio, rate, proportion and percent. A ratio shows a comparison of two or more quantities. A rate is a ratio that compares a quantity to 1 unit. When two ratios are equal, they form a proportion. A ratio of a number to 100 is called a percent. Percent thus simply means per hundred.

Learning outcome(s)
By the end of the unit, the participant will be able to

- explain the concepts of ratio and rate;
- solve real life problems using the concepts of ratio and rate;
- explain the concept proportion and express two ratios as a proportion;
- solve real life problems involving the concept of proportion;
- explain the concept percent and express ratios and fractions as percent;
- solve real life problems involving the concept of percent.

SESSION 1: RATIO AND RATE

We are aware of some of our traditional unequal ways of sharing among children. Some may be of the form *whenever the elder in the family takes 3, the younger child is to take 1,* or, *Mother takes 2 while father takes 3 or vice versa*. These are examples of the concept of ratio. The term **ratio** is derived from the Latin word '**ratus'** which means "to think" or "estimate". A ratio compares two quantities of the same dimension and unit. This means that, ratios involve comparing things. When we use ratio to compare two quantities we not only compare the same sort of things, (e.g. lengths with lengths, number of objects with number of objects) but we also use the same unit. When the two amounts in a ratio represent different quantities, we often refer to such ratios as **rates**. This session introduces you to the concept of ratio and how to solve simple problems involving ratio and rate.

Learning outcomes

By the end of the session, the participant will be able to explain the concepts of ratio and rate;

Concept of Ratio

One way of comparing two similar quantities is by expressing one as a fraction of the other. For example, if we compare the length of a stick 3cm long to another stick of length 4cm, we can write this as $\frac{3}{4}$. This fraction $\frac{3}{4}$ is the ratio of the two lengths. This can also be written as 3:4. The concept of ratio is a comparison by division. A ratio is a relationship between two amounts or quantities of the same units. Ratio is the relative sizes of two numbers, usually expressed as a:b or $\frac{a}{b}$, where $b \neq 0$. A ratio can be written in the following equivalent ways: using a fractional bar

($\frac{a}{b}$) or a colon ($a:b$). For example, the ratio of 3 to 5 can be written as $\frac{3}{5}$ or $3:5$. Both forms are useful.

The following examples give an idea that in ratio we compare objects with the same dimensions and units.

Example 1

If Kofi is 4 years old and Ama is 12 years old, we can say that the ratio of their ages is 4:12 or 1:3 or simply $\frac{1}{3}$. A ratio is always expressed in its simplest form.

Example 2

If there are 14 boys and 21 girls in a class, the ratio of the number of boys to the number of girls is 14 to 21, which can be expressed as the fraction $\frac{14}{21}$ which reduces to $\frac{2}{3}$. Similarly, we say the ratio of girls to boys is 21:14 or $\frac{21}{14}$ or $\frac{3}{2}$. We can also compare the number of boys to the number of pupils in the class to get 21:35 or $\frac{21}{35}$. The ratio of the number of girls to the number of pupils in the class to get 14:35 or $\frac{14}{35}$.

Example 3

If baby Atsu is 9 months old and his elder sister Peace is 3 years old, we do not say that the ratio of their ages is 9:3. Rather, we would change both ages into the same units.

The ratio is either 9 months: 36 months or, $\frac{3}{4}$ year: 3 years.

These are both equivalent to 1:4.

Ratios are usually written using two whole numbers.

Example 4

Mary's netball team won 16 games out of 24 games played in a season. What is the ratio of number of games won to number of games played?

Solution

Games won: games played $= 16:24$ or $\frac{16}{24}$.

This simplifies to $2:3$ or $\frac{2}{3}$.

Task: Now try the following question:

The records at a hospital on reported cases of malaria last month revealed that out of 360 patients admitted, 120 were children under 12 years, 180 were teenagers and the rest were adults. Determine the ratio of:

a. children to teenagers;
b. teenagers to adults;
c. children to adults;
d. children to teenagers to adults.

Check to see whether your answers are the following:

a. 120:180 or 2:3 or $\frac{2}{3}$.
b. 189:60 or 3:1 or $\frac{3}{1}$.
c. 120:60 or 2:1 or $\frac{2}{1}$.
d. 120:180:60 or 2:3:1

Try to explain your answers.

Concept of Rate

A ratio that compares a quantity to 1 unit is a rate. A **rate** is the comparison of one measure with another of a different dimension. The unit of rate is the number of units of the first quantity per one unit of the second quantity. For example, 30 miles per gallon or 60 kilometres per hour,

3 pineapples for 200 cedis, etc. In each case, one sort of unit is divided by another: "miles by gallon" or "kilometres by hours" or "pineapples by cedis". If a household consumes 240 units of electricity in 30 days, then the consumption rate is $\frac{240}{30}$ or 8 units per day.

In mathematics, a rate is a ratio between two related quantities in different units. In describing the units of a rate, the word "per" is used to separate the units of the two measurements used to calculate the rate. For example, a heart rate is expressed in "beats per minute". The most common type of rate is "per unit of time" such as speed and heart rate. Other examples are interest rates, discount and commission. Ratios that have a non-time denominator include exchange rates, literacy rates and electric field (in volts/meter).

Often rate is a synonym of rhythm or frequency, or count per second (i.e. Hertz); e.g. radio frequency or heart rate.

If the rate for any two pairs of two different quantities are the same, then the rate is said to be **uniform**. For example, Sarah is buying yoghurt for her best friend's birthday party. She buys a pack of 20 of the yoghurts for GH¢50.00. Sarah is wondering how much 1 pack of the yoghurt costs.

We can find the cost of one pack by dividing the cost/amount GH¢50.00 by the number of packs, which is 20. That is, $\frac{50}{20}$ which gives GH¢2.50.

Therefore, she buys 1 pack of yoghurt for GH¢2.50.

Example 5

A chemical shop received a consignment of 2,200 packs of a certain type of drugs. If the shopkeeper paid GH¢228,800.00 for the sale of the drugs, what is the price per pack?

Solution

Price per pack = amount paid divided by number of packs.

That is, $\frac{228,800}{2200} = 104$.

Therefore, the shopkeeper paid GH¢104.00 for one pack of the drug. That is, GH¢104.00 per pack.

Example 6

A driver covered a distance of 250km in two and half hours. Find the rate.

Solution

Rate $= \frac{\text{distance}}{\text{time}} = \frac{250km}{2\frac{1}{2}hrs} = 100\text{km/h}$ or 100 kilometres per hour.

Key ideas

- A ratio is a relationship between two amounts or quantities of the same units.
- It can be expressed as a:b or $\frac{a}{b}$, where $b \neq 0$.
- A **rate** is the comparison of one measure with another of a different dimension.
- The unit of rate is the number of units of the first quantity per one unit of the second quantity. For example, 30 miles per gallon or 60 kilometres per hour,
- If the rate for any two pairs of two different quantities are the same, then the rate is said to be **uniform**.

Reflection

1. Ethelinda has GH¢1,600.00 and Michael has GH¢2,400.00. what is the ratio of Michael's money to Ethelinda's money?
2. In a class are 24 boys and 32 girls. Find the ratio of the number of:

a. boys to girls;
b. girls to boys;
c. boys to the number of pupils in the class;
d. girls to the number of pupils in the class.

3. Determine the ratio of 40cm to 2m.
4. Express the following as ratios in the form a:b.
a. 18kg to 30kg;
b. 30 minutes to 3 hours;
c. 1,200m to 3 kilometres
5. Sule bought 3kg of meat for GH¢72.00. What is the cost per kilogram?
6. An ambulance covered a distance of 340 kilometres in 2 hours. What is the rate at which the ambulance travelled?
7. Seth earns GH¢960.00 for working for 8 hours. How much does he earn per hour?
8. Desire paid GH¢2,250.00 for 150 bottles of drinks. What is the cost of one bottle of drink?

SESSION 2: APPLICATION OF RATIO AND RATE

When people engage in joint business they often share the profits made according their contributions to the investment. Also, people make wills for their children that do not involve equal sharing of their wealth among the children. Such sharing is often done in a specified ratio.

This session discusses some applications of ratio and rate.

Learning outcomes

By the end of the session, the participant will be able to apply the concepts of ratio and rate to solve real life problems.

Let us consider the following problem:

A man shares his annual salary to his two sons, Kojo and Kwesi in the ratio $4:5$ respectively.

If the total annual salary to be shared is GH¢27,500.00,

a. how much did
i. Kojo receive?
ii. Kwesi receive?
b. How much more did Kwesi receive that Kojo?

Solution

a. (i) Ratio of Kojo' share: Kwesi's share $4:5$

Total ratio parts $=9$

Kojo's share $=\frac{4}{9}\times 27{,}500$

$=4\times 3000$ = GH¢12, 000.00

(ii) Kwesi's share $=\frac{5}{9}\times 27000$ $=5\times 3000$ = GH¢15,000.00

b. Difference in ratio is $5-4=1$

c. Difference in amount is $=\frac{1}{9}\times 27{,}500$ = GH¢3,000.00.

Or, GH¢15,000.00 - GH¢12,000.00 = GH¢3,000.00

Example 6

Authentia, and Elikem invested GH¢40,000.00 and GH¢30,000.00 respectively in a joint business venture. At the end of the first year of operation they realized a profit of GH¢2,100.00. They agreed to share the profit in the ratio of their contributions. How much of the profit did Authentia receive than Elikem?

Solution

Authentia's share :Elikem's share = 40000:30000. This simplifies to 4:3

Total number of parts 4 + 3 =7.

Authentia's share is $\frac{4}{7} \times 2100$ = GH¢1,200.00

Elikem's share is $\frac{3}{7} \times 2100$ = GH¢900.00

Or, GH¢2,100.00 - GH¢1,200.00 = GH¢900.00

Therefore, Authentia received GH¢1,200.00 - GH¢900.00 - GH¢300.00 more than Elikem.

Example 7

Mrs. Kabutey shared an amount of GH¢1,600.00 among her three children, Abena, Tawiah and Katey in the ratio 1:2:5.

a. Find the amount received by
   i. Abena;
   ii. Tawiah

b. How much did Katey receive than Abena?

Solution

Total number of ratio parts = 1 + 2 + 5 = 8

a. (i) Abena's share $\frac{2}{8} \times 1600$, which simplifies to GH¢200.00

(ii) Tawiah'sshare $\frac{2}{8} \times 1600$, which simplifies to GH¢400.00

b. Katey's share $\frac{5}{8} \times 1600$ = GH¢1,000.00

Therefore, Katey receive GH¢1000.00 - GH¢200.00 = GH¢800.00 more than Abena.

Alternatively, difference in ratios for Katey and Abena is 5 – 1 = 4.

Therefore, Katey had $\frac{4}{8} \times 1600$ = GH¢800.00 more than Abena.

Example 8

A forex bureau quoted the exchange rate for a particular day to be 5 Ghana cedis to 1 dollar and 6 Ghana cedis to 1 British pound.

a. Kofi has GH¢6,000.00 to be exchanged. How much would he receive if he wants
   i. Dollars ($);
   ii. British pounds (£).

b. Peterson came back from America with 550 pounds (£550) and $200. What is the total amount in cedis would he receive in exchange for the two currencies?

Solution

a. (i) Rate is GH¢5 to $1. Then GH¢6,000.00 will exchange for $\frac{6,000.00}{5}$ = $1,200.

(ii) Rate is GH¢6 to £1. Then GH¢6,000.00 will exchange for $\frac{6,000.00}{6}$ = £1,000.

b. Rate is £1 to GH¢6. Then £550 will exchange for 550×6 = GH¢3,300.00
Rate is $1 to GH¢5. Then $200 will exchange for 200×5 = GH¢1,000.00
Total amount is GH¢3,300.00 + GH¢1,000.00 = GH¢4,300.00

If the **rate** of any two parts of different quantities are not the same, then the rate is said to be **non-uniform**. The speed of a car travelling for any long distance journey is usually non-uniform. The rates for household utility bills and payment of taxes are also non- uniform.

Example 9

In Ghana, the annual income tax payable by an individual in a particular year is assessed at the following rates

First GH¢200.00 ------ free

Next GH¢300.00 ------ 50Gp per every GH¢50.00

Next GH¢300.00 ----- 100Gp per every GH¢50.00

Next GH¢300.00 ----- 200Gp per every GH¢50.00

Next GH¢300.00 ----- 400Gp per every GH¢50.00

Next GH¢300.00 ---- 500Gp per every GH¢50.00

Next GH¢300.00 ----- 600 Gp per every GH¢50.00

Andoh, an employee in a firm receives annual income of GH¢1,800.00.

How much does he pay as tax in a year?

Solution

Annual salary = GH¢1,800.00

Non- taxable income = GH¢200.00

Taxable income = GH¢ (1800-200) = GH¢1,600.00

Tax on 1st GH¢300.00 of remaining GH¢1,600 at 50Gp per every GH¢500.00

$= \frac{300 \times 0.5}{50} =$ GH¢3.00

Tax on next GH¢300.00 of remaining GH¢1,300 at 100Gp per every GH¢50.00

$= \frac{300 \times 1.00}{50} =$ GH¢6.00

Tax on next GH¢300.00 of remaining GH¢1,000 at 200Gp per every GH¢50.00

$= \frac{300 \times 2.00}{50} =$ GH¢12.00

Tax on the next GH¢300.00 of remaining GH¢700 at 400Gp per every GH¢50.00

$= \frac{300 \times 4.00}{50} =$ GH¢24.00

Tax on the next GH¢300.00 of remaining GH¢400 at 500Gp per every GH¢50.00

$= \frac{300 \times 5.00}{50} =$ GH¢30.00

Tax on the remaining GH¢100.00 at 600Gp per every GH¢50.00

$= \frac{100 \times 6.00}{50} =$ GH¢12.00

Total tax paid = GH¢[3+6+12+24+30+12] = GH¢87.00

Key ideas

- The concepts of ratio and rate can be applied in solving real life problems.
- For example, when people engage in joint business they often share the profits made according their contributions to the investment.

Reflection

1. A cash prize of GH¢15,000.00 was shared between two winners, Mensah and Otchere in the ratio 6:9. What is the share of each of the winners?
2. Serwaa, Forkuo and Damba shared an amount of GH¢8,400.00 in the ratio 1:4:7. Find the share of each person.
3. Share 12,000 packs of drugs among three hospitals, A, B, and C, in the ratio 3:5:7 respectively and indicate the share for each hospital.
4. It take 4 hours to bake cakes. Assuming no change of rate, how many hours will take 8 cooks to bake 8 cakes? (Answer 4 hours)

5. Jojo started school at 6 years when the mother Zashie was 28 years old.
a. What is the ratio of the age of Jojo to the age of the mother?
b. What would be the ratio of their ages when :
i. Jojo completed senior high school at age 15?
i. Jojo completed Nurses Training at age 19.

6. Sena and Danso contributed sums of money to the church collection in the ratio 3:5 respectively. If Danso contributed GH¢6,000.00, calculate the total amount contributed by the two friends.
7. The domestic power tariffs or charges for homes by the Electricity Corporation for Ghana for the amount of electricity used in a particular period were as follows:

First 50 units used ------ GH¢4.00 fixed
Next 100 units used ---- GH¢1.20 per every unit consumed.
Next 150 units used ------ GH¢1.50 per every unit consumed
Additional units used ----- GH¢2.20 per every unit consumed
Government Special levy ------ GH¢0.17 per every unit consumed
Street lights ----- GH¢0.05 per every unit consumed

i. In a chargeable period, a household consumed 550 units. Calculate the amount paid by the household during the period.
i. How much does a household pay for 800 units consumed?

8. Three people shared an amount of GH¢300,000.00 in the ratio 1:2:3. Find the highest amount received. [**Answer** GH¢**159,000.00]**
9. A tin of rice is consumed by n boys in 8 days, when three more boys joined them, the rice lasts only 6 days. Find n, if the rate of consumption is uniform. [**Answer n = 9]**
10. A bus travels a distance of 56km at an average speed of 70km/h. It travels a further 60km at an average speed of 50km/h.
Calculate, for the whole journey.
i. the total time taken **[Answer 2hrs]**
ii. the average speed. [**58km/h]**

SESSION 3: CONCEPT OF PROPORTION

We have learnt that ratio is used to compare two or more quantities of the same unit or dimensions. We also learnt that in ratio one number can be expressed as a fraction of the second number and then simplify. We know that $\frac{3}{5}$ and $\frac{12}{20}$ are equal ratios since $\frac{12}{20} = \frac{3 \times 4}{5 \times 4}$ which simplifies to $\frac{3}{5}$. We refer to the statement $\frac{3}{5} = \frac{12}{20}$ as a proportion. This session discusses the concept of proportion.

Learning outcomes

By the end of the session, the participant will be able to express two equal ratios as a proportion.

When two ratios are equal, they then form a proportion. A **proportion** is an equation whose members are ratios. The statement $\frac{3}{5} = \frac{12}{20}$ can also be written as $3:5 = 12:20$.

Proportions are ratios, but whereas ratios are used to compare similar quantities and numbers and have no units, proportions are used to compare quantities with different units. Proportions therefore, involve two equivalent ratios or rates.

The numerators and denominators of the ratios are called **terms** of the proportion. The terms of a proportion are numbered. In the generalized proportion $\frac{a}{b}=\frac{c}{d}$ or $a:b=c:d$, a is the first term, b is the second term, c is the third term and d is the fourth term. The first term and the fourth term of a proportion are called the **extremes** and the second and third terms are called the **means**.

One way to determine whether two ratios form a proportion is to check their cross products. Every proportion has two cross products: the numerator of one ratio multiplied by the denominator of the other ratio. If the cross products are equal, a proportion is formed. Thus, in a proportion, the product of the means is always equal to the product of the extremes, that is, $a\times d=b\times c$. By using this property of proportions, it is possible to solve a proportion for and any one of the terms, given the other three.

Example 10

Determine whether the fractions $\frac{27}{36}$ and $\frac{54}{72}$ form a proportion.

We write each fraction in the simplest form.

That is, $\frac{27}{36}=\frac{3\times 9}{4\times 9}=\frac{3}{4}$

And $\frac{54}{72}=\frac{3\times 18}{4\times 18}=\frac{3}{4}$.

Since each ratio/fraction simplifies to the same fraction $\frac{3}{4}$ they form a proportion.

Alternatively, we find the product of extremes and the product of the means and then check if they are the same. That is, we cross multiply.

We write $\frac{27}{36}=\frac{54}{72}$.

Then we cross multiply to get $27\times 72=1944$ and $54\times 36=1944$. Since the two products give the same number 1944, the fractions form a proportion.

Example 11

Determine whether the ratios 32:56 and 12:21 form a proportion.

Solution

We write each ratio in the simplest form.

The ratio 32:56 has 8 as a common factor to the numbers and so reduces to 4:7.

The second ratio 12:21 has 3 as a common factor to the numbers and so reduces to 4:7.

Since the two ratios simplify to the same ratio, we say ratios 32:56 and 12:21 form a proportion and we write $32:56 = 12:21$.

Alternatively, we can express the two ratios in fractional form and then simply each.

That is, $\frac{32}{56} = \frac{4\times 8}{7\times 8} = \frac{4}{7}$ and $\frac{12}{21} = \frac{4\times 3}{7\times 4} = \frac{4}{7}$. Thus each ratio simplifies to the same fraction $\frac{4}{7}$ and so they form a proportion.

Example 12

Solve for x if $\frac{3}{5} = \frac{x}{20}$.

Solution

We form the cross products and equate them since $\frac{3}{5} = \frac{x}{20}$ is a proportion.

Product the extremes is ($3\times 20)$ and product of the means is $(5x)$.

Equating the products we have $5x = 3\times 20$

Dividing through by 5 we have $x = \frac{3\times 20}{5} = 12$.

Therefore, x = 12; that is $\frac{3}{5} = \frac{12}{20}$

Example 13

Find the value of m if $3:m = 12:108$.

Solution

Product of extremes = product of means.

That is, $m\times 12 = 3\times 108$

$$m = \frac{3\times 108}{12} = 27$$

Therefore, $m = 27$; that is, $3:27 = 12:108$

Key ideas

- A **proportion** is an equation whose members are ratios.
- The numerators and denominators of the ratios are called **terms** of the proportion.
- In the generalized proportion $\frac{a}{b} = \frac{c}{d}$ or $a:b = c:d$, a is the first term, b is the second term, c is the third term and d is the fourth term.

- The first term and the fourth term of a proportion are called the **extremes** and the second and third terms are called the **means**.

Reflection

1. Determine whether or not each of the following ratios form a proportion.

a. $5:6$ and $35:42$

b. $36:39$ and $24:26$

c. $5:12$ and $3:7$

d. $5:25$ and $20:100$

e. $\frac{6}{7}$ and $\frac{7}{8}$

f. $\frac{5}{6}$ and $\frac{55}{66}$

2. Find the value of *y* if $\frac{15}{35} = \frac{75}{y}$
3. Find the value of *x* if $\frac{4}{25} = \frac{6000}{x}$
4. What value of *k* makes the statement $\frac{144}{180} = \frac{k}{120}$ true?
5. Find the value of *p* if $60:p = 3:4$.
6. Find the value of *w* if $w:54 = 8:12$.

SESSION 4: APPLICATION OF PROPORTION

We notice that if we know the cost of say, one book to be GH¢8.00, then the cost of two such books is GH¢16.00, the cost of three is GH¢24.00; and we realize that as we buy more such books we pay more amounts which are in multiples of 8. We can write the rates and proportions and then write corresponding proportions. The rates are 1 book: GH¢8; 2 books:GH¢16; 3 books: GH¢24 and so on. The corresponding proportions are $1:8 = 2:16$; $1:8 = 3:24$; $2:16 = 3:24$.

The proportions from the ratios are $1:2 = 8:16$; $1:3 = 8:24$; $2:3 = 16:24$. This kind of situation is described as **direct proportion** whereby as one quantity increases, the second quantity also increases and vice versa. There are also situations in which as one quantity increases the second quantity rather decreases and vice versa. This situation is referred to as **indirect proportion**. This session deals with some application of proportions involving direct and indirect proportions.

Learning outcomes

By the end of the session, the participant will be able to solve real life problems involving the concept of proportion.

Direct Proportion

Suppose 20kg of cooking oil costs GH¢75.00. What will be the cost of 36kg of the same cooking oil? We know that the cost of more kilograms of cooking oil will also increase. We can solve this by forming the ratios and then forming the corresponding proportion. We first let the cost of the required (36kg) be *y*. Then the corresponding proportion is

$20kg : 36kg =$ GH¢75: $\text{GH¢}75 : \text{GH¢}y$

Writing this in fractional form gives $\frac{20}{36} = \frac{75}{y}$.

We apply the cross multiplication rule and get

$$y = \frac{36 \times 75}{20} = 135$$

We conclude that the cost of 36kg of oil is GH¢135.00.

Example 14

A manufacturer packed 3600 packets of drugs into 10 boxes. Find into how many boxes of the same size 5400 packets could be packed?

Solution:

Unitary Method

In this approach we first find the number of packets one box can contain.

If 3600 packets are in 10 boxes,

then one box contains $\frac{3600}{10} = 360$

Therefore, 5400 packets will go into $\frac{5400}{360} = 15$ boxes

Or,

Let n be the number of boxes in 5400 packets, then the proportion is

3600 packets: 5400 packets = 10 boxes: n boxes

$$3600 : 5400 = 10 : n$$

Therefore, $n = \frac{5400 \times 10}{3600} = 15$ boxes

Example 15

If the property tax on a GH¢65,000.00 house is GH¢240.00, what is the tax on anGH¢85,000.00 house?

Solution

The proportion is GH¢65,000:GH¢85,000 = GH¢240:*n*

Product of extremes = product of means gives

$$65000 \times n = 85000 \times 240$$

$$n = \frac{85000 \times 240}{65000} \cong 313.85$$

Therefore, the tax on the GH¢85,000.00 house is GH¢313.85.

Example 16

In a particular season, 1,500 people consumed 900 kilograms of meat. How many kilograms of meat should be ordered for 1,800 such people?

Solution

The proportion is 1,500:1,800 = 900:K

In fractional form we have $\frac{1500}{1800} = \frac{900}{K}$

Cross multiplying, $1500K = 1800 \times 900$

And $K = \frac{1800 \times 900}{1500} = 1080$

Therefore, 1,080kg of meat should be ordered.

Indirect Proportion

Two or more quantities are said to be an indirect (inverse) proportion if one quantity increases as the other decreases or vice versa. For example,

If a certain piece of work can be done in 10 days by 4 men, then it can be done in 5 days by 8 men, 2 days by 20 men and so on. The time taken varies inversely as the number of men. Alternatively, it may be said that the time taken varies as the reciprocal of the number of men. The reciprocal of 5 is 15, the reciprocal of 34 is 43, and the reciprocal of 65 is 56. In general, the reciprocal of the fraction abis ba.

Task: Write down the reciprocal of each of the following fractions: 49,712,83,112, and 15.

Example 17

If 18 workers harvest corn on a farm in 6 days, how many workers would finish harvesting the same amount of corn in 4 days, working at the same rate?

Solution

Method 1

We can see here that we need more workers to finish the work in 4 days.

More members needed in few days and less workers needed for more days keeping at the same rate. If 18 workers harvest in 6 days, then 4 days would require

$\Rightarrow \frac{18 \times 6}{4} = 27$ workers

Method 2

18 people worked in 6 days implies that 1 person takes 18×6days

Therefore, number of workers for 4 days =18×64=27 days

Example 18

A bag contains sweets. When divided amongst 8 children, each child receives 9 sweets. If the sweets were divided amongst 12 children, how many sweets would each child receive?

Solution

Notice that because there are more children, we would expect each child to receive less than before (i.e. less than 9). We could say:

"8 children received 9 sweets each, that makes a total of 72 sweets. Divide this between 12 and each child would get 6 sweets."

Or,

The ratio of children is 12:8 = 3:2, therefore, the ratio of the sweets should be 2: 3. Two thirds of 9 is 6. Using reciprocal, 812=n9, then n=8×912=6 sweets

Example 19

Eight boys cleared a piece of farmland in 6 hours. How long will it take 4 boys to clear the same piece of farmland at the same rate?

Solution

If 8 boys used 6 hours, then 4 boys will use longer time.

If 8 boys used 6hrs then

4 boys will use $= \frac{8}{4} \times 6 = 12$ hours

Key ideas

- The kind of situation whereby as one quantity increases, the second quantity also increases and vice versa is described as **direct proportion.**
- The situations in which as one quantity increases the second quantity rather decreases and vice versa is referred to as **indirect proportion**.

Reflection

1. Madam Ablewor shared novels between her two children Kojo and Aseewaa in direct proportion to their ages. If Kojo who is 12 years old received 6 books, find the number of books Aseewaa received if she is 14 years old. **(14 books)**
2. Amenu can type 450 words in 6 minutes. How long does he take to type 1200 words? (**16 minutes**)
3. If 8 bottles of soft drink cost GH¢ 20.00, how much will 24 bottles of the drink cost? (GH¢60.00)
4. An Assembly man donated book prizes as awards for the two best mathematics JHS3 students in the constituency. The awards were given in direct proportion to their scores in the mathematics test. Ethelinda scored 91 marks and got 7 books. How many books does Abudu receive as a prize if he scored 78? **(6 books)**
5. On a given map, 5cm length corresponds to 20km on land, what is the distance on land if it is represented by 12cm on the map? (**48km)**
6. If 4 boys can clear a piece of land in 15 hours, how long does it take 5 boys to clear the same piece of land if all are working at the same rate? (**12 hours**)
7. If 6 packs of a certain type of drug cost GH¢90.00, find the cost of 20 packs of the same drug.
8. Sixteen men working $7\frac{1}{2}$ hours a day can do a piece of work in 6 days. How long will 9 men take, working 8 hours a day to do the same piece of work? (**10 days**)
9. A car travelling at 120km/h covered a distance in 5 hours. What is the speed of a car that takes 4 hours to cover the same distance?
10. It takes 30 people working at the same rate to complete a job in 8 days. How many people will complete the same job in 12 days if they work at the same rate?

SESSION 5: CONCEPT OF PERCENT

A percent is a ratio whose second term is 100. Because the second term is always the same i.e. 100, percent offers a convenient way to compare two quantities.

'Percent' is made up of two words: 'per' meaning 'divide by' or 'out of' and 'cent' meaning 'hundred'. The symbol %, read "per cent' is used to indicate a percent. For example 25 percent thus means '25 out of 100' and written 25%.

Percentages are special ratios which compare a number of parts with 100 parts (1 whole).

This session deals with the concept of percent and how to express the various forms ratios as percent.

Learning outcomes

By the end of the session, the participant will be able to explain the concept percent and express ratios and fractions as percent.

Converting a Percentage as a Fraction

To convert a percentage as fractions, one needs to write the given percent as fraction with denominator 100 (i.e. Divide the given percent by 100) and simplify the resulting fraction if possible.

Example 20

Express 25% as fraction.

Solution

Convert 25% as a fraction with denominator 100, i.e. $25\% = \frac{25}{100}$

Simplify the resulting fraction, i.e. $\frac{25}{100} = \frac{1}{4}$

Example 21

Express 17.5% as a fraction.

Solution

Write 17.5% as a fraction with denominator 100, $\Rightarrow 17.5\% = \frac{17.5}{100}$.

Simplify the resulting fraction, $\Rightarrow \frac{17.5}{100} = \frac{35}{200} = \frac{7}{40}$.

Converting a Fraction as Percent

Converting a fraction as a percent can be done by writing the given fraction and multiplying it by 100% and then simplify the result, or, forming proportions with an unknown and solve.

Example 22

Express $\frac{3}{4}$ as a percent.

Method I

Multiply $\frac{3}{4}$ by 100%, to get

$$\frac{3}{4} \times 100\% = \frac{300}{4}\% = 75\%$$

Method II

Form proportions by first introducing a variable.

Lets x be the required percent. Then,

$x : 100 = 3 : 4$ or $\frac{x}{100} = \frac{3}{4}$

Solving, $4x = 300$

$x = 75$

Example 23

Express $\frac{7}{20}$ as a percent.

Solution

Method I

Multiply by 100%. $\frac{7}{20} \times 100\% = \frac{700}{20} = 35\%$

Method II

Using proportions
Let y be the required percent.
Then $\frac{y}{100} = \frac{7}{20}$
$\Rightarrow 20y = 7 \times 100$
$\Rightarrow \quad y = 35\%$

You can also change a fraction to a percent by using long division to change the fraction to a decimal and then change the resulting decimal to a per cent.

Example 24

Change $\frac{7}{8}$ to a per cent.

Solution

Use long division to change $\frac{7}{8}$ to decimal.

$8\sqrt{7.000}$

$$8\overline{)7.000}^{\,0.875}$$

Task: Now change 0.875 to a per cent. The answer is 87.5%.

Decimal to Percentages

To change a decimal to a per cent, we can

1. Express the decimal as a fraction whose denominator is 100; and
2. Rewrite the fraction as a percent.

Example 25

Change 0.37 to a per cent.

Solution

Writing decimal as a fraction with denominator 100.

$0.37 = \frac{37}{100}$

Now rewriting $\frac{37}{100}$ as a per cent is 37%.

After becoming familiar with this kind of problems many people omit the fraction step and simply move the decimal point, 2 places to the right and add % symbol when changing a decimal to a per cent. This is the same as multiplying the given decimal fraction by 100. Hence

a. 0.17 = 17%
b. 0.027 = 2.7%
c. 0.01 = 1.0%
d. 4.67= 467%

To change a per cent to a decimal

1. Replace the percent with a fraction whose denominator is 100;
2. Divide the numerator by 100

Example 26

Change 36% to a decimal

Solution

Rewrite 36% as $\frac{36}{100}$. Then divide 36 by 100.

That is $\frac{36}{100} = 0.36$

Again, with familiarization, you may omit the fraction step. You simply move the decimal point two places to the left and drop the % symbol when changing from a per cent to a decimal.

Key ideas

- A percent is a ratio whose second term is 100.
- Percentages are special ratios which compare a number of parts with 100 parts (1 whole).
- The symbol %, read "per cent' is used to indicate a percent. For example 25 percent thus means '25 out of 100' and written 25%.

Reflection

1. Express the following fractions as percentages
a. $\frac{3}{25}$ b) $\frac{19}{20}$ c) $\frac{3}{5}$ d) $\frac{5}{8}$ e) $\frac{17}{40}$
2. Change each of the following percentages to a fraction:
a) 42% ($\frac{21}{50}$) b) 54% c) 24% d) 66% e) 75% f) 98% g) 0.45%
3. Convert each of the following decimal fractions as a fraction:
a. 0.25 b) 0.36 c) 0.85 d) 0.95 e) 0.025 f) 0.005

SESSION 6: APPLICATION OF PERCENTAGES

In the previous session, we discussed the concept of percent and how to change various ratios to percentage and vice versa. We have varied applications of percentages in every aspect of our daily activities, such as Increase and Decrease, Profit and Loss, Discounts, Commissions, Simple and Compound Interests. This session deals with some applications of percentages.

Learning outcomes

By the end of the session, the participant will be able to solve real life problems involving the concept of percent.

Basic Percent Problems

There are three types of basic per cent problems:

1. Finding a per cent of a given quantity (number).
2. Finding what per cent one quantity (number) is of another quantity (number).
3. Finding a quantity (number), given a per cent of that quantity (number).

Example 27

What is 36% of 75?

Solution

Let $x =$ the number sought

Change the percent to a fraction or a decimal and rewrite the word problem as an equation.

$$x = \frac{36}{100} \times 75 = 0.36 \times 75 = 27$$

Example 28

Given that 12% of a given number is 48 what is the number?

Solution

Method I

i. Let x be the number
ii. Change the percent to a fraction or decimal and rewrite the word problem as an equation

$$\Rightarrow \frac{12}{100} \times x = 48$$ or

$$\Rightarrow 0.12x = 48$$

iii. Solve the resulting equation

$$\Rightarrow \frac{12}{100} \times x = 48$$

$$\Rightarrow 12x = 4800$$

$$\Rightarrow x = 400$$

Or simply, $0.12x = 48$

And $x = 400$

Method II

i. Let x = the number

ii. Change the per cent to a fraction and rewrite the word problem as a proportion

1. $\frac{48}{x} = \frac{12}{100}$

iii. Solve the resulting proportion

$$\frac{48}{x} = \frac{12}{100}$$
$$12x = 4800$$
$$x = 400$$

Example 29

Sixteen is a certain per cent of 80. Find the percent.

Solution

Method I

i. Let x = the per cent sought

ii. Change the per cent to a fraction and rewrite the word problem as an equation

$$16 = \frac{x}{100} \times 80$$

iii. Solve the resulting equation

$$16 = \frac{x}{100} \times 80$$
$$1600 = 80x$$
$$x = 20$$

Method II

i. Let $x =$ the percent

ii. Change the percent to a fraction and rewrite the word problem as a proportion

1. $\Rightarrow \frac{x}{100} = \frac{16}{80}$

iii. Solve the resulting proportion

$$\Rightarrow \frac{x}{100} \times \frac{16}{80}$$
$$80x = 1600$$
$$x = 20$$

Example 30

A school library purchased 10 new books last month. Of the books purchased, 20% were mathematics books. How many were mathematics books?

Solution

Let n = the number of math books

$$20\% \Rightarrow \frac{20}{100} \times 10 = n$$

$0.2 \times 10 = n$
$n = 2$ books
The number of math books is 2.

Example 31

There are 500 pupils in a primary 6 class. A survey showed that 15% of them take the bus to school. How many of these pupils do not take the bus to school?

Solution

Total number of pupils = 500

Number of pupils who take the bus $= \frac{15}{100} \times 500 = 75$

Therefore number of pupils who do not take bus to school are $500 - 75 = 425$

Or,

Percent of pupils who do not take the bus $\Rightarrow (100 - 15)\% = 85\%$

Therefore, the number of pupils who do not take the bus to schools $= \frac{85}{100} \times 500 = 425$

Expressing a Quantity as a Percentage of Another Quantity

We can express one quantity as a percentage of another quantity of the same unit.

Example 32

Express GH¢145.00 as a percentage of GH¢950.00.

Solution

In solving a problem such as this, we must understand that the quantity which comes (is stated) after the word **'of'**, is always expressed as the denominator and the other quantity as the numerator to form a fraction which is multiplied by 100%.

$$\Rightarrow \frac{GH¢145}{GH¢950} \times 100\% = 15.26\% \quad \approx 15.3\%$$

Percentage Increase or Decrease

We can increase or decrease a quantity by a certain percent value.

To increase a quantity A by a certain percent x, can be done in two ways.

The first approach is to calculate x percent of A and then add this to A to get the required amount. The second is to represent the original quantity A by 100% and the new quantity by $(x + 100)\%$, and then find $(x + 100)\%$ of A.

Method I

i. Calculate x percent of A

ii. Add this to A to get the required amount.

I.e. $[\frac{x}{100} \times A] + A = A(1 + \frac{x}{100})$

Method II

i. Original quantity A = 100%

New quantity = $(x + 100)\%$

ii. Find $(x + 100)\%\ of\ A$

$$\Rightarrow \frac{(100 + x)}{100} \times A$$

Example 33

Increase GH¢550.00 by 20%

Method I

Solution

$\frac{20}{100} \times 550 = GH$¢110.00

New amount $= GH$¢[550+110] = GH¢660.00

Method II

Original amount GH¢550.00 = 100%

New amount $(100 + 20)\% = 120\%$

New amount $= \frac{120}{100} \times 550 = GH$¢660.00

Example 34

The cost of a shirt was GH¢65.00 in January 2019. It was later increased by 15% in July 2019. What is the new cost of the shirt?

Solution

Original cost of shirt $= (100 + 15)\% = 115\%$ of original cost

New cost $= \frac{115}{100} \times 65 = GH¢74.75$

To Decrease a Quantity by a Percentage x

Original quantity =100%

New value $= (100 - x)\%$ of the original

Or calculate $x\%$ of original quantity and subtract from the original quantity. i.e. $[A-(\frac{x}{100}\times A)] = A[1-\frac{x}{100}]$

Example 35

Decrease 162m of the length of a room by 42%.

Solution

Method I

Original length of room = $162cm \Rightarrow 100\%$.

New length of room = $(100-42)\% = 58\%$

New length of room = $\frac{58}{100}\times 162cm = 93.96cm$

Method II

Decrease in length = $\frac{42}{100}\times 162cm = 68.04cm$

Therefore new length of room = $(162-68.04)cm = 93.96cm$

Profit and Loss

In the business of buying and selling of items, profit and/or loss may occur. **Cost Price (CP)** is the price at which an article is put up on sale or the price tag the article has on the shop. It is also simply known as the shopkeeper's price. **Selling (Sale) Price (SP)** is the price at which the customer buys (pays to the shopkeeper) for the article or item. **Profit** is the difference between the Selling price (SP) and the Cost price (CP). **Loss** occurs if the trader cannot sell the item (article) at a price higher than the cost price, but lower than the cost price, then the difference between the cost price and the selling price is called '**loss'**.

Profits or Losses are expressed as ratios to the cost price.

Net Profit = Selling Price – Cost Price or SP – CP

Percentage Profit or Loss = $\frac{Profit / Loss}{Cost\ Price}\times 100\%$

Cost Price = Selling Price – Profit or Loss

$$Selling\quad Price = \frac{100 + profit\ \%}{100}\times CP$$

$$Cost\quad Price = \frac{100}{100 + profit\ \%}\times SP$$

Example36

Ama bought 50 pineapples at GH¢1.50 each. She sells all the pineapples at GH¢1.80 each. How much profit did she make?

Solution

Method I

Total cost of 50 pineapples to Ama at GH¢1.50 each = 50 x GH¢1.50 = GH¢75.00.
Total selling price for 50 pineapples at GH¢1.80 each = 50 x GH¢1.80 = GH¢90.00.
Profit made = GH¢ [90 – 75] = GH¢15.00

Method II

Cost price for each pineapple = GH¢1.50
Selling price for each pineapple = GH¢1.80
Profit on each pineapple = GH¢[1.80 – 1.50] = GH¢0.30
Total profit made on 50 pineapples = 50 x 0.30 = GH¢15.00

Example 37

Kojo bought 1,000 plastic chairs for hiring business for GH¢1,250.00 after a year in business, he sold them for GH¢1,055.00 at a loss. Calculate his loss percent.

Solution

Cost price = GH¢1,250.00 and Selling price = GH¢1,055.00
Loss = CP – SP
= GH¢ [1250 – 1055] = GH¢195.00

Loss % = $\frac{195}{1250} \times 100\% = 15.6\%$

Discounts

At times, shops give discounts to try to encourage customers to buy in larger quantities or pay cash instead of using cheques or buying on credits. The discount is usually quoted as a percentage of the price of the article. Cash discount is discount allowed for prompt payment. Net price is the price after discount has been deducted. The price quoted by the shop is called the Marked or Catalogued price.

Example 38

An article marked GH¢250.00 is subject to a discount of 5% for cash payments made by a customer. Find the cash price of the article.

Solution

Method I

Discount $= \frac{5}{100} \times 250 = GH¢12.50$

Cash (Net) price $= GH¢[250 - 12.50] = GH¢237.50$

Method II

Marked price = 100% = GH¢250.00

Cash (Net) price = (100 – 5)% = 95%

$$= \frac{95}{100} \times 250 = GH¢237.50$$

Example 39

A customer paid GH¢450.00 for an item after a 10% discount had been allowed from the listed price. What is the listed price of the item?

Solution

Discount = 10% of the listed price
Price paid = (100 – 10)% of the listed price = GH¢450.00
Listed price = 100% of the price paid

$$= \frac{100}{90} \times 450 = GH¢500.00$$

Commission

Commission is a percentage payment made by one person to another person in consideration of service rendered or for purchase or sale of goods on his behalf. It is simply a fee paid an agent who is involved in the financial transactions and is often calculated as the percentage of the amount of money involved.

Example 40

A sales representative receives a salary of GH¢250.00 and a commission of 5% on all sales over the first GH¢2,500.00 of sales made. If her sales in a month was GH¢3250.00, what was her income?

Solution

Salary = GH¢250.00

Sales over first GH¢2,500.00 = GH¢ [3250 – 2500] = GH¢750.00

Total income = Commission + Salary

$$= [\frac{5}{100} \times 750] + 250$$

= 37.50 + 250
= GH¢287.50

Example 41

A trader received a commission of 12.5% on sales made in a month. If for a month, he made a total sales of GH¢25,550.00, find his commission.

Solution

Commission = 12.5% of GH¢25,550.00

$$= \frac{12.5}{100} \times 25550 = GH¢3,193.75$$

Simple Interest

Interest is a charge for the use of money. Most people at some time in their life will use a bank. They may have their wages and salaries paid directly to the bank and they may save money in the bank. Sometimes, they will wish to borrow money from the bank. If they save money with the bank, they will find that the bank 'pays' them to do so. Each year, they leave their money in the bank, the bank will increase the amount by a certain percentage. This extra payment is called '**Interest'** and the percentage paid is called the '**Interest rate'.**

There are two kinds of interest namely: Simple interest and Compound interest. We shall discuss these in details.

There are three factors that influence the simple interest calculations:

- The Principal (P), which is the amount of money save or borrowed
- The Rate (R), which is the percentage value at which the interest is agreed
- The Time (T), which gives the duration for saving or borrowing the money.

The **'Amount', A,** at the end of the period is calculated by:

A = Principal + Interest.

The interest (I) on the principal (P) for the time (T), in years is proportional to the product of P and T. i.e. $I \propto PT$

$\therefore I = constant \times PT$, where Rate (R) is the constant.

$\Rightarrow I = PTR$

Since the rate is always expressed in percentage (%), the interest is expressed as:

$$I = \frac{PTR}{100} \quad \text{.............................(1)}$$

Note that we can have

$$Rate(R) = \frac{100}{PT} \quad \text{..........................(2)}$$

$$Time(T) = \frac{100}{PR} \quad \text{............................(3)}$$

$$Principal(P) = \frac{100}{TR} \quad \text{...........................(4)}$$

Example 42

Kofi invested an amount of GH¢560.00 at the bank at simple interest rate of $7\frac{1}{2}$ % per annum for 4 years. Calculate his

i. Interest at the end of the period
ii. Amount at the end of the 4 years.

Solution

P = GH¢560.00, T = 4 years, R = $7\frac{1}{2}$ %

i. Interest (I) = $\frac{560 \times 4 \times 7\frac{1}{2}}{100} = \frac{560 \times 4 \times 15}{200} = GH¢78.40$

ii. Amount (A) = Principal + Interest
= GH¢ [560 + 78.40]
= GH¢638.40

Example 43

Mr. Boadi borrowed an amount of GH¢11,500.00 from a financial institution at 21% per annum simple interest. Calculate the amount of interest he would have to pay for a period of 3 years and what would be the total amount of money he has to pay back to the financial institution at the end of the 3 years.

Solution

Principal (P) = GH¢11,500.00
Rate (R) = 21%
Time (T) = 3 years

Interest (I) = $\frac{11500 \times 3 \times 21}{100}$
$= GH¢7245.00$

Amount at end of period = GH¢ [11,500 + 7245]
= GH¢18,745.00

$\therefore$ His total interest was GH¢7,245.00 and the amount at the end of the period was GH¢18,745.00.

Key ideas

- **Net Profit = Selling Price – Cost Price or SP – CP**
- **Percentage Profit or Loss =** $\frac{Profit/Loss}{Cost\ Price} \times 100\%$
- **Cost Price = Selling Price – Profit or Loss**
- $Selling\quad Price = \frac{100 + profit\ \%}{100} \times CP$
- $Cost\quad Price = \frac{100}{100 + profit\ \%} \times SP$
- Interest (I) = principal P×Time T×Rate (R)100
- $Rate(R) = \frac{100}{PT}$
- $Time(T) = \frac{100}{PR}$
- $Principal(P) = \frac{100}{TR}$

Reflection

1. What percentage of GH¢2,500.00 is GH¢1,750.00?
2. Madam Margaret imported a quantity of frozen fish costing GH¢400.00. The goods attracted an import duty of 15% of its cost. She also paid a sales tax of 10% of the total cost of the goods including the import duty and the sold the goods for GH¢660.00. Calculate for the percentage profit. [**Answer 30.43%]**
3. A container holds 5.5 litres of cooking oil. If 3.5 litres is used up, what percentage of the original volume is left? (36.36%)
4. In a college of 900 students, 350 are ladies and the rest are men. Find the percentage of men in the college. (61.1%)
5. Afia sold an article for GH¢5,700.00 in the market making a profit of 15%. Find the original price of the article. ($GH¢4,956.52$)
6. A shopkeeper bought a TV set for GH¢670.00 and resold it to a customer at a profit of 21%. What is the shopkeeper's selling price? ($GH¢810.70$)
7. Mansa paid an amount of GH¢180.00 for a dress when a discount of 10% was allowed. Find the marked price of the dress. ($GH¢200.00$)
8. The marked price of a dress is GHC95.00. A discount of 10% is allowed for cash payments made. How much does one save by paying cash for the dress? ($GH¢9.50$)
9. A property sales agent receives a 15% commission on any property sales made up to GH¢15,000.00. In addition, he is given a commission of 7% on any sales over the first GH¢15,000.00. How much does he receive as commission if in a month his total sale was GH¢30,500.00? (**GH¢3,335.00**)
10. A wholesaler allowed a trade discount of 15% on the catalogued price of a refrigerator. If the catalogued price of the refrigerator was GH¢3,250.00, how much did the retailer pay? ($GH¢2,762.50$)
11. Simon saved an amount of money at the bank for 5 years at simple interest rate of 17% per annum. At the end of the period he was paid an amount of GH¢45,325.00. How much did he save at the bank initially? ($GH¢24,500.00$)
12. Kofi bought a TV set at $GH¢695.00$ last year. The cost of the same TV set had been reduced by 17.5% this year. What is the new cost of the TV set? ($GH¢121.63$)
13. In Ghana, the annual income tax payable by an individual in a particular year is assessed at the following rates

First GH¢200.00 ------ free
Next GH¢300.00 ------ 50Gp per every GH¢50.00
Next GH¢300.00 ----- 100Gp per every GH¢50.00
Next GH¢300.00 ----- 200Gp per every GH¢50.00
Next GH¢300.00 ----- 400Gp per every GH¢50.00
Next GH¢300.00 ---- 500Gp per every GH¢50.00

Next GH¢300.00 ----- 600 Gp per every GH¢50.00

A worker in a firm receives annual income of GH¢2,000.00.

i. How much does he pay as tax in a year?
i. What percentage of his income was payable as income tax?

UNIT 4: NUMBER BASES AND THEIR APPLICATIONS

In this unit, we involve students in activities, which will enable them to count objects and write numbers in various number bases, perform basic operations. We will also provide worthwhile tasks on simple equations involving number bases for students to solve.

Learning outcome(s)

By the end of the unit, the participant will be able to:

1. Convert numerals from other bases to base ten.
2. Convert numerals from base ten to other bases.
3. solve number bases up to 12.
4. Apply number bases.
5. Solve equations involving number bases

Definition

A number base is the number of digits or combination of digits that a system of counting uses to represent numbers. A base can be any whole number greater than 0. The most commonly used number system is the decimal system, commonly known as base 10. The set of digits for base ten is {0,1,2,3,4,5,6,7,8,9}. So *93 in base ten* means 9 *tens* and 3 *ones*. We may sometimes also count in other bases. For example, 35 in base 7 (written as 35_7 or 35_{seven}) means 3 sevens and 5 ones. The base two numeration system is known as the decimal system while the base 12 numeration system is known as the duodecimal system.

Converting Numerals from Other Bases to Base Ten

In order to change a numeral written in other bases to base ten, we rewrite the numeral as an expanded numeral in base ten and then simplify.

Example 44

Convert 567_{eight} to a number in base ten.

Solution

- rewrite the numeral as an expanded numeral in base ten and write the place values starting from the right-hand side.

$$567\text{eight}=5\times82+6\times81+(7\times80)$$

- multiply each digit by its base raised to the corresponding place value

$$567\text{eight}=5\times64+6\times8+(7\times1)$$

$$567\text{eight}=320+48+(7)$$

- add up the products.

$$567\text{eight}=375\text{ten}$$

Example 45

Convert 13233four to a decimal numeral

Solution

$13233\text{four}=1\times44+3\times43+2\times42+3\times41+(3\times40)$

13233four=1×256+3×64+2×16+3×4+(3×1)

13233four=256+192+32+12+(3)

13233four=495ten

Example 46

Convert 1101.101two to a decimal numeral

Solution

1101.101two=1×23+1×22+0×21+1×20+1×2-1+0×2-2+(1×2-3)

1101.101two=8+4+0+1+0.5+0+0.125

1101.101two=13.625ten

Converting Numerals from Base Ten to Other Bases

A number write in base ten van be converted to a numeral in another base by continually division. This is known as the method of repeated division.

Example 47

Convert 1059ten to a number in base six

6	1059		
6	176	*rem*	3
6	29	*rem*	2
6	4	*rem*	5
	0	*rem*	4

Solution

1059÷6=176 remainder 3

176÷6=29 remainder 2

29÷6=4 remainder 5

4÷6=0 remainder 4

Therefore, 1059ten is 4523six

Example 48

Convert 375ten to base four

Solution

375÷4=93 remainder 3

93÷4=23 remainder 1

23÷4=5 remainder 3

5÷4=1 remainder 1

1÷4=0 remainder 1

4	375		
4	93	*rem*	3
4	23	*rem*	1
4	5	*rem*	3
4	1	*rem*	1
	0	*rem*	1

Therefore, 375ten is 11313four

Bases Greater Than Ten

In order to operate in a number base greater than ten, it is important to create extra symbols or digits. For example base twelve has 12 symbols and may be listed as 0,1,2,3,4,5,6,7,8,9,***T,E.*** (Where ***T*** and ***E*** represent ten and eleven respectively).

Example 49

Convert 4T7Etwelve to base ten.

Solution

4T7Etwelve=4×123+T×122+7×121+(E×120)

4T7Etwelve=4×123+10×122+7×121+(11×120)

4T7Etwelve=6912+1440+85+11

4T7Etwelve=8447ten= 8447

Example 50

Convert 1436ten to a number in base twelve.

12	1436		
12	119	*rem*	8
12	9	*rem*	*E*
	0	*rem*	9

Solution

Therefore, 1436ten=9E812

Solving Equations Involving Number Bases

We can solve for the base of a number in an equation in base ten and then solve for *n*.

Example 51

Find the value of *n* in the following equations.

a. 34n=19
b. 61n=1335

Solution

a. 34n=19
34n=3×n1+(4×n0) = 19
3×n1+(4×n0) = 19
3n+(4×1) = 19
3n+4 = 19
3n = 19 – 4
3n = 15
n = 5

b. 61n=1335
6×n1+(1×n0) = 1×52+3×51+(3×50)
6n+(1) = 1×25+3×5+(3×1)
6n+1 = 25+15+3
6n = 42
n = 7

Reflection

1. Convert the following to base five
a. 675ten
b. 1456
c. 325712
2. Find the value of x in each of the following
a. 123x=11
b. 3324=222x

UNIT 5: INDICES AND LOGARITHMS

In this unit, we introduce Indices and Logarithms, which is central to many scientific and business processes. If we consider our daily activities such as finance, economics, agriculture, etc. we use the concept of indices and logarithms unknowingly. So in this we take a look at the fundamentals of laws governing indices and logarithms and their expressions. We will also discuss how to solve equations involving indices and logarithms.

Learning outcome(s)

By the end of the unit, the participant will be able to:

1. state and justify the laws of indices
2. work with derived laws of indices
3. solve single variable equation involving indices
4. define logarithms using indices
5. state and proof the laws of logarithms
6. solve single variable equation involving logarithms
7. combine the concepts of indices and logarithms to solve equations.

SESSION 1: DEFINITION AND LAWS OF INDICES (FUNDAMENTAL LAWS)

In this session the concept of indices by looking at the definition and the fundamental laws that governs the operations of indices.

Learning outcomes

By the end of the session, the participant will be able to:

1. define indices
2. state and proof the laws of indices

Definition

Indices is a number of the form a^m where *a* is called the base and *m* is the index (also called the exponent or power).

We note that the indices a^m is also call a power and it is said to be the product of a certain number factors, all of which are the same. For example, the indices 3^5 has the number 3 to be the five factors of the number 3^5, all of which are the same. Here 3 is the base and 5 is the index (exponent or power). Numbers can be express in their power form, e.g. 8 has its power form to be 2^3, 9 has its power to be 3^2, etc. This power can also be written in an expanded form to illustrate the repeated multiplication of factors which are all the same. For example $3^7 = 3\times3\times3\times3\times3\times3\times3$, likwise $a^5 = a\times a\times a\times a\times a.$

Laws of Indices

Law 1: Multiplication Law

This law is given as

$$a^m \times a^n = a^{m+n}.$$

For example $a^4 \times a^3 = a^{4+3} = a^7$. This can alternatively be expressed as

$$\begin{aligned} a^4 \times a^3 &= (a \times a \times a \times a) \times (a \times a \times a) \\ &= a \times a \times a \times a \times a \times a \times a \\ &= a^7. \end{aligned}$$

Law 2: Division (Quotient) Law

This law is given as

$$a^m \div a^n = a^{m-n}.$$

For example $a^5 \div a^3 = a^{5-3} = a^2$. This can alternatively be expressed as

$$\begin{aligned} a^5 \div a^3 &= \frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} \\ &= a \times a \\ &= a^2. \end{aligned}$$

Remark

We note that in the two laws the bases of the indices being multiplied or divided must be the same.

Law 3: Power Law

This law is given as

$$(a^m)^n = a^{mn}.$$

For example $(a^4)^3 = a^{4\times 3} = a^{12}$. This can alternatively be expressed as

$$\begin{aligned} (a^4)^3 &= (a \times a \times a \times a)^3 \\ &= (a \times a \times a \times a) \times (a \times a \times a \times a) \times (a \times a \times a \times a \times a) \\ &= a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \\ &= a^{12}. \end{aligned}$$

Example 52

Simplify the following:

(a) $2^3 \times 2^5$ (b) $2^6 \times 2^2 \times 2$ (c) $x^7 \div x^3$ (d) $3^5 \div 3^7$

Solution

$$\begin{aligned} \text{(a) } 2^3 \times 2^5 &= 2^{3+5} \quad \text{(Product Law)} \\ &= 2^8. \end{aligned}$$

(b) $2^6 \times 2^2 \times 2 = (2^6 \times 2^2) \times 2$

$= 2^{6+2} \times 2$ (Product Law)

$= 2^8 \times 2$

$= 2^{8+1}$ (Product Law)

$= 2^9$

(c) $x^7 \div x^3 = x^{7-3}$ (Division Law)

$= x^4$.

(d) $3^5 \div 3^7 = 3^{5-7}$ (Division Law)

$= 3^{-2}$.

Key ideas

- Indices is a number of the form a^m where *a* is called the base and *m* is the index (also called the exponent or power).
- **Laws of Indices**
  - $a^m \times a^n = a^{m+n}$.
  - $a^m \div a^n = a^{m-n}$.
  - $(a^m)^n = a^{mn}$.

Reflection

Simplify the following expressions

1. $a^7 \times a^2$	2. $a^5 \times a^4 \times a^3$	3. $b^5 \times b^{-3} \times b^6$	4. $b^4 \times b^{11} \times b^{-8}$
5. $a^5 \div a^2$	6. $2^{-4} \div 2^3$	7. $5^{13} \div 5^{21}$	8. $x^{5a} \div x^{3a}$.

SESSION 2: DERIVED LAWS OF INDICES

In this session, we will consider deriving some other laws of indices from the product, division and power laws that we have already learnt.

Learning outcomes

By the end of the session, the participant will be able to:

- state the derived laws.
- use the derived laws.

Law 4: Zero Power Law

This law can be written as

$$a^0 = 1.$$

In other words we can simply put as any number to the power of zero is 1. This can be derived from the division law as follows:

We know that

$$a^m \div a^n = a^{m-n}.$$

If we let $m = n,$ then we have

$$\begin{aligned} a^m \div a^m &= a^{m-m} \\ &= a^0. \end{aligned}$$

But

$$\begin{aligned} a^m \div a^m &= \frac{a^m}{a^m} \\ &= 1 \end{aligned}$$

$$\therefore a^0 = 1.$$

Law 5: Negative Power Law

This law can be written as

$$a^{-m} = \frac{1}{a^m}.$$

We can also derive this law from the division as follows:

$$a^4 \div a^5 = \frac{a^4}{a^5} = \frac{a \times a \times a \times a}{a \times a \times a \times a \times a} = \frac{1}{a}.$$

But

$$a^4 \div a^5 = a^{4-5} = a^{-1}$$

$$\therefore a^{-1} = \frac{1}{a}.$$

$$a^4 \div a^7 = \frac{a^4}{a^7} = \frac{a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a} = \frac{1}{a^3}.$$

But

$$a^4 \div a^7 = a^{4-7} = a^{-3}$$

$$\therefore a^{-3} = \frac{1}{a^3}.$$

It follows that in general

$$a^{-m} = \frac{1}{a^m}.$$

Law 6: Fractional Power Law

This law can be written as

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

This means that any number raised to the power $\frac{1}{n}$ is equal to the n^{th} root of the number.

For example, $9^{\frac{1}{2}} = \sqrt[2]{9}$ and $8^{\frac{1}{3}} = \sqrt[3]{8}.$ We know the answers are 3 and 2 respectively. We however use the laws 1 and 3 to obtain these answers.

We perform the operations as follows:

$$9^{\frac{1}{2}} = (3\times 3)^{\frac{1}{2}} = (3^2)^{\frac{1}{2}} = 3^{2\times\frac{1}{2}} = 3.$$

$$8^{\frac{1}{3}} = (2\times 2\times 2)^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = 2.$$

We note that this same law can be written as

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = (\sqrt[n]{a^m}).$$

Law 7: Same Power Law

So far we have dealt with laws that have terms with common base. We will now consider laws to be used when the bases of the terms are not the same, but the powers are the same.

This law can be written as

$$a^n \times b^n = (a\times b)^n = (ab)^n$$

and

$$a^n \div b^n = (a \div b)^n = \left(\frac{a}{b}\right)^n.$$

Example 53

a. $\left(\frac{2}{3}\right)^{-3}$ b. $\frac{2^3\times 2^{-5}}{2^{-2}}$ c. $\frac{(a^2)^3}{b^3} \div \left(\frac{a}{b^2}\right)^{-2}$ d. $27^{\frac{2}{3}}\times 4^{-\frac{5}{2}}$

Solution

a. $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}.$

b. $\dfrac{2^3 \times 2^{-5}}{2^{-2}} = \dfrac{2^{3-5}}{2^{-2}} = \dfrac{2^{-2}}{2^{-2}} = 2^{-2} \times 2^2 = 2^{-2+2} = 2^0 = 1$

c. $\dfrac{(a^2)^3}{b^3} \div \left(\dfrac{a}{b^2}\right)^{-2} = \dfrac{a^{2\times 3}}{b^3} \div \left(\dfrac{b^2}{a}\right)^2 = \dfrac{a^6}{b^3} \div \dfrac{b^{2\times 2}}{a^2} = \dfrac{a^6}{b^3} \times \dfrac{a^2}{b^4} = \dfrac{a^6 \times a^2}{b^3 \times b^4} = \dfrac{a^8}{b^7}$

d. $27^{\frac{2}{3}} \times 4^{-\frac{5}{2}} = \left(\sqrt[3]{27}\right)^2 \times \left(\sqrt[2]{4}\right)^{-5} = 3^2 \times 2^{-5} = 9 \times \dfrac{1}{2^5} = \dfrac{9}{32}$.

Key ideas

- **Derived Laws of Indices**
  - $a^0 = 1$.
  - $a^{-m} = \dfrac{1}{a^m}$.
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$.
  - $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = (\sqrt[n]{a^m})$.
  - $a^n \times b^n = (a \times b)^n = (ab)^n$ and $a^n \div b^n = (a \div b)^n = \left(\dfrac{a}{b}\right)^n$.

Reflection

Simplify the following expressions

1. $\dfrac{a \times a^5 \times a^7}{a^3 \times a^6}$
2. $7^4 \times 49^{10}$
3. $\left(\dfrac{a^4 \times a^6}{a^3 \times a^7}\right)^{\frac{1}{2}}$
4. $8^{\frac{5}{3}}$
5. $\dfrac{(2a^{\frac{1}{2}})^3}{4a^2}$
6. $\left(\sqrt[4]{\dfrac{81}{64}}\right)^{\frac{2}{3}}$
7. $a\left(2a^{-\frac{1}{4}}\right)^4$
8. $\dfrac{(a^{2m})(a^{m+2})}{a^{3m}}$

SESSION 3: EQUATIONS INVOLVING INDICES

In this session, we will apply the laws of indices we learnt in sessions 1 and 2 to solve equations involving indices. We will consider only equations with variable exponents.

Learning outcomes

By the end of the session, the participant will be able to:

1. Solve equations involving indices.

To solve equations involving indices, we first make sure that all the terms in the equation have common base, then we equate the exponents and solve for the unknown variable.

Example 54

Solve the following equation for *x*.

a. $2^x = 1$ b. $25^x = 125$ c. $4^{x-3} = 64$ d. $4^{3x} = 32^{x+1}$ e. $8^{x+3} = \left(\frac{1}{2}\right)^{x+1}$.

Solution

a. $2^x = 1$

$2^x = 2^0$

Equate exponents

$\therefore\ x = 0.$

b. $25^x = 125$

$(5^2)^x = 5^3$

$5^{2x} = 5^3$

Equating exponents we obtain

$2x = 3$

$\therefore \quad x = \frac{3}{2}.$

c. $4^{x-3} = 64$

$(2^2)^{x-3} = 2^6$

$2^{2(x-3)} = 2^6$

Equating exponents we obtain

$2(x-3) = 6$

$x - 3 = 3$

$\therefore \quad x = 6.$

d. $4^{3x} = 32^{x+1}$

$$(2^2)^{3x} = (2^5)^{x+1}$$

$$2^{6x} = 2^{5(x+1)}$$

Equating exponents we obtain

$$6x = 5(x+1)$$

$$6x = 5x + 5$$

$$\therefore \quad x = 5.$$

e. $8^{x+3} = \left(\frac{1}{2}\right)^{x+1}$

$$(2^3)^{x+3} = (2^{-1})^{x+1}$$

$$2^{3(x+3)} = 2^{-1(x+1)}$$

Equating exponents we obtain

$$3(x+3) = -1(x+1)$$

$$3x + 9 = -x - 1$$

$$4x = -10$$

$$\therefore \quad x = \frac{-10}{4} = -\frac{5}{2}.$$

Reflection

Solve the following equation for *x*.

1. $6^{-x} = 216$ 2. $49^x = \frac{1}{343}$ 3. $2^{x+2} = 0.125$ 4. $\frac{8}{2^{-x}} = 32^x$

5. $3^{4x} \div 3^{3-x} = \frac{1}{243}$ 6. $2^{x+5} \div 4^{-x} = 32$ 7. $(3^x)^3 \times 3^{3-x} = \frac{1}{27}$ 8. $2^{2x+3} - 2^{2x+1} = \frac{3}{8}$.

SESSION 4: DEFINITION LOGARITHMS

Having studied indices in the previous sessions, this session is the appropriate place for logarithms because a logarithm is an index. In future the student will realize that logarithms are very useful for multiplication and division and some of the properties are useful in more advanced work.

Learning outcomes

By the end of the session, the participant will be able to:

1. define logarithm in terms of the base and the index.
2. convert indices to logarithms and vice versa.

Definition

Consider the number 1000, this can be express as a power of 10, that is $1000 = 10 \times 10 \times 10 = 10^3$. In indices we have 1000 to be 10 raised to the power (index or exponent) 3. In logarithms we say that exponent 3 is the logarithm of 1000 to base 10. In general if have the equation $x = a^n$, we say that the **exponent *n* is the logarithm of *x* to base *a***. That is, $n = \log_a x$ if and only if $x = a^n$. The following examples will exclusively show the conversion between indices and logarithms.

Example 55

Change each of the following index forms into their equivalent logarithmic forms.

a. $2^4 = 16$ b. $27 = 3^3$ c. $7^3 = 343$ d. $125^{\frac{1}{3}} = 5$ e. $0.0001 = 10^{-3}$ f. $1 = 10^0$.

Solution

a. $2^4 = 16 \Rightarrow 4 = \log_2 16$

b. $27 = 3^3 \Rightarrow 3 = \log_3 27$

c. $7^3 = 343 \Rightarrow 3 = \log_7 343$

d. $125^{\frac{1}{3}} = 5 \Rightarrow \frac{1}{3} = \log_{125} 5$

e $0.0001 = 10^{-3} \Rightarrow -3 = \log_{10} 0.0001$

f. $1 = 10^0 \Rightarrow 0 = \log_{10} 1$

Example 56

Change each of the following logarithmic forms into their equivalent index forms.

a. $\log_2 32 = 5$ b. $2 = \log_a x$ c. $\log_{10} 10 = 1$ d. $\log_{10} 0.1 = -1$ e. $\log_{2.6} 6.76 = 2$ f. $\log_e 0.3Q = -t/RC$

Solution

a. $\log_2 32 = 5 \Rightarrow 32 = 2^5$

b. $2 = \log_a x \Rightarrow x = a^2$

c. $\log_{10} 10 = 1 \Rightarrow 10 = 10^1$

d. $\log_{10} 0.1 = -1 \Rightarrow 0.1 = 10^{-1}$

e. $\log_{2.6} 6.76 = 2 \Rightarrow 6.76 = 2.6^2$

f. $\log_e 0.3Q = -t/RC \Rightarrow 0.3Q = e^{-t/RC}$

Key ideas

- Generally in the equation $x = a^n$, we say that the **exponent *n* is the logarithm of *x* to base *a***. That is, $n = \log_a x$ if and only if $x = a^n$.
- By convention logarithm to base 10 $(\log_{10})$ is called a **common logarithm** and logarithm to base *e* ($\log_e$ or sometimes written as $\ln$) is called a **natural logarithm**.
- The natural logarithm is mostly preferred in scientific and advanced studies in mathematics.

Reflection

1. Express the following equation in logarithmic form

   a. $10^4 = 10000$ b. $7^0 = 1$ c. $64 = 16^{\frac{3}{2}}$ d. $0.001 = 10^{-3}$ e. $81 = (1/3)^{-4}$

   f. $a^5 = b$ g. $p^q = r$ h. $1 = x^0$ i. $16^{-1/4} = 1/2$ j. $0.000001 = 10^{-5}$

2. Express in index notation

   a. $\log_{23} 23 = 1$ b. $\log_3 9 = 2$ c. $\log_{10} 2x = 0.1$ d. $\log_4 2 = 1/2$ e. $\log_e 1 = 0$

   f. $-2 = \log_3 1/9$ g. $3 = \log_a b$ h. $u = \log_v w$ i. $-1 = \log_{10} 0.1$ j. $-2 = \log_{10} 1/100$

3. Evaluate

   a. $\log_{10} 100$ b. $\log_8 2$ c. $\log_2 64$ d. $\log_{10} 10^3$ e. $\log_e 1/e$ f. $\log_{2/3} 4/9$

SESSION 5: LAWS OF LOGARITHMS

In this session we state the laws of logarithms and used the laws of indices to deduce them.

Learning outcomes

By the end of the session, the participant will be able to:

1. state the laws of logarithms.
2. deduce the laws using laws of indices.
3. use the laws to solve mathematics problems involving logarithms.

Law 1: Product Law

This law is given as

$$\log AB = \log A + \log B.$$

Deduction

Let $A = 10^x$ and $B = 10^y$ in terms of logarithms we have $\log_{10} A = x$ and $\log_{10} B = y$.

Now consider

$$A \times B = 10^x \times 10^y$$

$$AB = 10^{x+y}.$$

Expressing this equation in logarithmic form, we obtain

$$\log_{10} AB = x + y.$$

By substituting $\log_{10} A$ for x and $\log_{10} B$ for y, we obtain

$$\log_{10} AB = \log A + \log B.$$ **As required**.

Law 2: Quotient Law

This law is given as

$$\log A/B = \log A - \log B.$$

Deduction

Let $A = 10^x$ and $B = 10^y$ in terms of logarithms we have $\log_{10} A = x$ and $\log_{10} B = y$.

Now consider

$$A \div B = 10^x \div 10^y$$

$$A/B = 10^{x-y}.$$

Expressing this equation in logarithmic form, we obtain

$$\log_{10} A/B = x - y.$$

By substituting $\log_{10} A$ for x and $\log_{10} B$ for y, we obtain

$$\log_{10} A/B = \log A - \log B.$$ **As required**.

Law 3: Power Law

This law is given as

$$\log A^n = n \log A.$$

Proof

Let $A = 10^x$ in terms of logarithms we have $\log_{10} A = x.$

Now consider

$$A^n = 10^{nx}$$

Expressing this equation in logarithmic form, we obtain

$$\log_{10} A^n = nx.$$

By substituting $\log_{10} A$ for x, we obtain

$$\log_{10} A^n = n \log A.$$ **As required**.

Example 57

Express the following as single logarithms:

a. $\log 2 + \log 5$,   b. $\log 28 - \log 7$,   c. $\log 4 + 2\log 3 - \log 6$.

Solution

a. We know that $\log AB = \log A + \log B$

$\Rightarrow \log 2 + \log 5 = \log(2 \times 3) = \log 6.$

b. We know that $\log A/B = \log A - \log B$

$\Rightarrow \log 28 - \log 7 = \log(28 \div 7) = \log 4.$

Example 58

Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, evaluate the following logarithms:

a. $\log_{10} 6$,   b. $\log_{10} 15$,   c. $\log_{10} 600$.

Solution

a.
$$\begin{aligned}\log_{10} 6 = \log_{10}(2 \times 3) &= \log_{10} 2 + \log_{10} 3 \\ &= 0.3010 + 0.4771 \\ &= 0.7781\end{aligned}$$

b.
$$\begin{aligned}\log_{10} 15 = \log_{10}(3 \times 5) &= \log_{10} 3 + \log_{10} 5 \\ &= \log_{10} 3 + \log_{10} 10/2 \\ &= \log_{10} 3 + \log_{10} 10 - \log_{10} 2 \\ &= 0.4771 + 1 - 0.3010 \\ &= 1.1761\end{aligned}$$

c.
$$\begin{aligned}\log_{10} 600 = \log_{10}(2 \times 3 \times 100) &= \log_{10} 2 + \log_{10} 3 + \log_{10} 10^2 \\ &= \log_{10} 2 + \log_{10} 3 + 2\log_{10} 10 \\ &= 0.3010 + 0.4771 + 2 \times 1 \\ &= 2.7781\end{aligned}$$

Example 59

Simplify:

a. $\frac{1}{2}\log_3 81$,   b. $\frac{1}{3}\log 8$,   c. $\frac{\log 8}{\log 9}$.

Solution

a. $\frac{1}{2}\log_3 81 = \log_3 81^{\frac{1}{2}}$

$= \log_3 9$

$= \log_3 3^2$

$= 2\log_3 3$

$= 2.$

b. $\frac{1}{3}\log 8 = \log 8^{\frac{1}{3}}$

$= \log 2.$

c. $\frac{\log 81}{\log 9} = \frac{\log 3^4}{\log 3^2}$

$= \frac{4\log 3}{2\log 3}$

$= 2.$

Key ideas

- **Laws Of Logarithms**
  - $\log AB = \log A + \log B.$
  - $\log A/B = \log A - \log B.$
  - $\log A^n = n\log A.$
- Readers should note that where the base is omitted it is assumed to be a common logarithm.
- Facilitators are encourage to teach students how to use scientific calculators to evaluate common and natural logarithms.

Reflection

1 Express the following as single logarithms:

a. $\log 18 - \log 9$, b. $\log b + \log c$ c. $3\log 2 + 2\log 3 - 2\log 6$, d. $5 - 2\log x$.

2 Given that $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 5 = 0.6990$ and $\log_{10} 7 = 0.8451$, find the value of the following:

a. $\log_{10} 14$, b. $\log_{10} 40$, c. $\log_{10} 1/6$, d. $\log_{10} 0.00042$.

3 Simplify

a. $\frac{1}{3}\log_2 64$, b. $\frac{1}{2}\log 36$, c. $5\log 2 - \log 32$, d. $\frac{\log 49}{\log 343}$.

SESSION 6: EQUATIONS INVOLVING LOGARITHMS AND CHANGE OF BASE

In this final session we will study two category of equation: the first will be equations that involve indices but it is not practical to express all the terms in the same base and will require the use of properties indices and logarithms combined; the second category are purely logarithmic equations with variables to be solved for.

Learning outcomes

By the end of the session, the participant will be able to:

1. Solve equations that require the combine use of the laws of indices and logarithms.
2. Change the base of a given logarithm.

Category I

This type of indicial equations it is not practical to be express all the terms in the same base. The way out in this case is to employ some laws of logarithms as well as use of a scientific calculator.

Example 60

Solve the following equations, leaving the answers to four decimal places.

a. $5^x = 7$, b. $7^{2x-1} = 11$, c. $2^{2x} = 5^{x-3}$.

Solution

a. $5^x = 7$ (Take logarithms to base 10 on both sides.)

$$\Rightarrow \log_{10} 5^x = \log_{10} 7$$
$$x \log_{10} 5 = \log_{10} 7$$
$$x = \frac{\log_{10} 7}{\log_{10} 5} \quad \text{(Use calculator to find the logarithms.)}$$
$$x = \frac{0.8451}{0.6990} = 1.2090.$$

$b.$ $7^{2x-1} = 11$

$$\Rightarrow \log 7^{2x-1} = \log 11$$
$$(2x-1)\log 7 = \log 11$$
$$2x - 1 = \frac{\log 11}{\log 7}$$
$$2x - 1 = \frac{1.0414}{0.8451}$$
$$2x - 1 = 1.2323$$
$$2x = 0.2323$$
$$\therefore \quad x = 0.1162$$

$c.$ $2^{2x} = 5^{x-3}$ (Take natural logarithms on both sides)

$$\ln 2^{2x} = \ln 5^{x-3}$$
$$2x \ln 2 = (x-3)\ln 5$$
$$\frac{2x}{x-3} = \frac{\ln 5}{\ln 2}$$
$$\frac{2x}{x-3} = \frac{1.6094}{0.6931}$$
$$2x = 2.3220(x-3)$$
$$2x = 2.3220x - 6.9660$$
$$-0.3220x = -6.9660$$
$$\therefore \ x = 21.6335.$$

Category II

This category, we will them logarithmic equations. In simple cases, we can convert a logarithmic equation into an indicial equivalent. In a more advanced and complex case, the problem is simplified such that there is only a term on both sides of the equation each expressed in logarithmic form to the same base. Once this done, then the property of logarithms are applied.

Example 61

Solve the following logarithmic equations, leaving the answers to four decimal places.

a. $\log_2(3x-1)=2$, b. $\log_9(5x-4)=\log_3 4$, c. $\log_5(x+2)+\log_5(x+4)=1$.

Solution

a. $\log_2(3x-1)=2$

Express this in index form

$$3x-1=2^2$$
$$3x-1=4$$
$$3x=5$$
$$\therefore \quad x=5/3.$$

b. $\log_9(5x-4)=\log_3 4$

We change base on the left side to base 3 (procedure change of bases).

$$\Rightarrow \quad \frac{\log_3(5x-4)}{\log_3 9}=\log_3 4$$
$$\frac{\log_3(5x-4)}{\log_3 3^2}=\log_3 4$$
$$\frac{\log_3(5x-4)}{2}=\log_3 4$$
$$\log_3(5x-4)=2\log_3 4$$
$$\log_3(5x-4)=\log_3 4^2$$

Let

$$\log_3(5x-4)=\log_3 4^2=w$$
$$\Rightarrow \log_3(5x-4)=w \text{ and } \log_3 16=w$$

Expressing each of them in index form, we have

$$5x-4=3^w \text{ and } 16=3^w.$$
$$\Rightarrow 5x-4=16$$
$$5x=20$$
$$\therefore x=4.$$

c. $\log_5(x+2)+\log_5(x+4)=1$

Combine the terms on the left side using the product law and let replace 1 with $\log_5 5$.

$\Rightarrow \log_5(x+2)(x+4)=\log_5 5$

let $\log_5(x+2)(x+4)=\log_5 5=w$

$\Rightarrow \log_5(x+2)(x+4)=w$ and $\log_5 5=w$

Express each in index form

$(x+2)(x+4)=5^w$ and $5=5^w$

$\Rightarrow (x+2)(x+4)=5$

$x^2+6x+8=5$

$x^2+6x+3=0$

$$\Rightarrow x=\frac{-6\pm\sqrt{24}}{2}$$

$$=\frac{-6\pm 2\sqrt{6}}{2}$$

$$=-3\pm\sqrt{6}$$

$$\therefore x=-3+\sqrt{6} \text{ or } x=-3-\sqrt{6}.$$

Change of Base

Let $\log_a x=c.$ Then, $a^c=x.$

Taking logarithm of each side to base b, we obtain

$$\log_b a^c=\log_b x$$

$$\Rightarrow c\log_b a=\log_b x$$

$$c=\frac{\log_b x}{\log_b a}$$

But $c=\log_a x.$ Hence

$$\log_a x=\frac{\log_b x}{\log_b a}.$$

Note, in a special case where $x=b,$ we obtain

$$\log_a b=\frac{\log_b b}{\log_b a}=\frac{1}{\log_b a}.$$

Reflection

1 1 Solve the equations:

a. $2^x = 5$,   b. $2^x \times 2^{x+1} = 10$,   c. $(1/2)^x = 6$,   d. $5^{2x-1} = 11$.

2 Solve the following logarithmic equations, leaving the answers to four decimal places.

a. $\log_{11}(x^2 - 6) = 1/3$,   b. $\log_2 x - \log_4 3 = 2$,

c. $\log_{10}(x^2 - 4x + 7) = \log_{10} 100$,   d. $\log_2 x - 3\log_x 2 = 2$.

UNIT 6: RELATIONS AND FUNCTIONS

In this unit, we introduce relations and functions and their types by way of definition and some concepts associated with them such as domain, co-domain, image and range. We will also consider how to operate on given functions using the four basic operations and graph these functions. It is important to know both how to operate on functions and particular features of their graphs.

Learning outcome(s)

By the end of the unit, the participant will be able to:
1. distinguish between relations and functions and identify the
2. various types of relations and functions;
3. distinguish among concepts used in functions namely domain, co-
4. domain, image and range;
5. add, multiply, subtract and divide functions;
6. determine the composite of given functions;
7. find the inverse of simple functions;
8. graph any given function.

SESSION 1: RELATIONS AND FUNCTIONS AND THEIR TYPES

In this session we will consider introducing relations and functions and their types. In doing so, we shall consider formal definitions of these two concepts used in mathematics and their differences.

Learning outcomes

By the end of the session, the participant will be able to:

1. give the definition of relations.
2. state the types of relations.
3. give the definition of functions.
4. state the types of functions.

Relation (Definition)

Consider that the price of a book is GH₵ 3.00. Then the amount you pay for purchasing books increases depending on the number of books you intend to buy. Similarly, the amount spent on airtime increases with time. In each case, it can be observed that one quantity (money) depends on another (books purchased or talking hours). This suggests a relationship exists between the two quantities. Now, let's turn our attention to what is meant by **relation**, mathematically, having explored the idea of relation informally.

A **relation** is a set of ordered pairs (x, y). There is a connection between one quantity (x) or a first set and a second quantity (y) such that *y* depends on *x.* Let *A*, *B* be two non-empty sets and *R* be a subset of $A \times B$. *R* is called a *relation* from A to B, if (*x*, *y*) R, then *x* is said to be in relation (*R*) to *y* written *x R y* (read *x* has the relation *R* to *y*). The set of *x* in *A* for each of which there is a *y* in *B* with (*x*, *y*) in *R* is called the *Domain* of *R*. The set of *y* in *B* for each of which there is an *x* in A with (*x*, *y*) in *R* is called the range of *R*.

Having looked at the definition of relation, now, let us look at the various types of relations.

Functions

A function is a relation with one additional condition. There are a number of ways of defining functions. One way of defining functions is as a relationship between an input and an output. Remember when you were in primary school and learned about operation machine?

An operation machine or function **machine** can explain how a function operates. In this definition, the idea of a function is a specially relationship between an input (*x*-value) and output (*y*-value where every input value has only one output value. If you put in a certain sequence of numbers, you will get out a definite sequence of numbers. Sometimes you can determine the rule by examining the input and output numbers.

Another way of defining functions is the use of sets. That is, given two sets, A and B, a function *f* is a collection of ordered pairs *(x, y)* where $x \in A$ read "*x belongs to A*" and $y \in B$ read "*y belongs to B*" and <u>every</u> element in *A* is associated with a <u>unique</u> element in *B* by the function *f*. That is, a function, *f* , from *A* to *B*, where *A* and *B* are non-empty sets, is a rule that associates, with each element of *A*, a unique element of *B*. The set *A* is the *domain* (D_f) and *B* is the *co-domain* of the function (*f*). In other words, a function is a set of ordered pairs such that <u>no pair has the same first element or component</u>. The notation; $f : A \to B,$ (read as '*f from A to B*') is used.

Can you explain what "*unique*" means? Hopefully, you wrote something like "one and only one"! In summary, a function is a set of ordered pairs (*x, y*) such that for any value of *x*, there is exactly one value of *y*. It is a rule that assigns to each element *x* in a set *A* exactly one element, called *f(x)* in a set *B.*

Worked Examples: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4, 5\}$. Which of the following sets of ordered pairs or figures represent functions from sets *A* to set *B*?

a. $\{(a, 2), (b, 3), (c, 4)\}$ b) $\{(a, 4), (b, 5)\}$

a. $\{(a, 2), (b, 3), (c, 4)\}$ is a function because each element of *A* is matched with exactly one element of *B.*

b. $\{(a, 4), (b, 5)\}$ is **not** a function because not every element of *A* is matched with an element of *B*

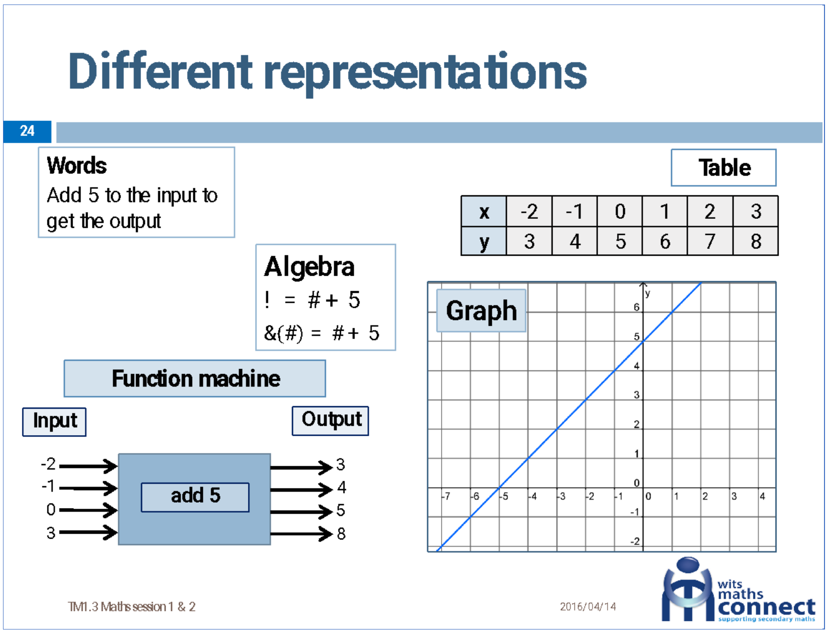
c. The arrow diagram in c) is a function from *A* to *B*. It does **not** matter that each element of *A* is matched with the same element of *B.*

d. The diagram does **not** represent a function from *A* to *B* because some elements in *A* are matched with two elements in *B*.

Ways of Representing Functions

For example, if input values $\{-1, 0, 1, 2, 3\}$ yield the following output values $\{-1, 1, 3, 5, 7\}$ respectively, then we get the ordered pairs $\{-1, 1\}$, $\{0, 1\}$, $\{1, 3\}$, $\{2, 5\}$ and $\{3, 7\}$. The rule for these ordered pairs is $f(x) = 2x + 1$ or $y = 2x + 1$. It is convenient to describe a function by giving a formula for its ordered pairs, what we call, the rule for the function. For example a function may be defined as $f(x) = 3x + 1, x \in \Re$ or $f : x \to 3x + 1, x \in \Re.$ (function notation)

There are five different ways of representing functions namely: 1) words; 2) table; 3) algebra (rule); 4) function machine; and 5) graph. See the figure below.



Source: WMCS -2019

We can describe a specific function in the following five ways:

i. verbally (by description in words; e.g. $C(x)$ is "the cost of producing x items")

ii. algebraically (by an explicit formula; e.g., $A(r) = \pi r^2$)

iii. visually (by a graph)

iv. numerically (by a table)

v. function machine (input and output / set notation).

Key ideas

- A **relation** is a set of ordered pairs (x, y).
- If *A and B* are two non-empty sets and *R* be a subset of $A \times B$. *R* is called a *relation* from A to B, if (*x, y*) R, then *x* is said to be in relation (*R*) to *y* written *x R y* (read *x* has the relation *R* to *y*).
- The set of *x* in *A* for each of which there is a *y* in *B* with (*x, y*) in *R* is called the *Domain* of *R.*
- The set of *y* in *B* for each of which there is an *x* in A with (*x, y*) in *R* is called the range of *R.*
- That is, given two sets, A and B, a function *f* is a collection of ordered pairs *(x, y)* where $x \in A$ read "*x belongs to A*" and $y \in B$ read "*y belongs to B*" and <u>every</u> element in *A* is associated with a <u>unique</u> element in *B* by the function *f*.
- In other words, a function is a set of ordered pairs such that <u>no pair has the same first element or component</u>.
- Functions may be represented as: 1) words; 2) table; 3) algebra (rule); 4) function machine; and 5) graph.

Reflection

1. Given the statement: “subtract 7 from the input to get the output”, represent this statement as follows:
    a. table;

b. graph

c. rule

d. function machine

2. Give an example of a relation that is **not** a function.

3. Which of the following arrow diagrams define function(s)? Explain your answer in each case.

SESSION 2: DOMAIN AND RANGE OF FUNCTIONS

When looking at the various definitions of functions, we started being introduced to some concepts that are used in functions namely: the domain and range. In this session, we shall look at how to determine the domain, co-domain, image and range of functions.

Learning outcomes

By the end of the session, the participant will be able to distinguish among concepts used in functions namely

1. domain
2. co-domain
3. image
4. range

Now read on…

Let us begin with an example. Given: $\{(-1, 0), (0, 0), (1, 2), (2, 1)\}$, then we can form the set of all the first values in the given ordered pairs as follows: $\{-1, 0, 1, 2\}$ which are also our input values (think about the operation/function machine) and the set of all second values in the ordered pairs as follows: $\{0, 1, 2\}$ also known as the output values .

The set of input values is the **domain** and the set of output values is called the **range**. We usually consider functions for which the sets *A* and *B* are sets of real numbers. The symbol $f(x)$ read “*f* of *x*” or “*f* at *x*” is called the **value of *f* at *x***, or the **image of *x* under *f***. The set *A* is called the **domain** of the function. In the given example, the set $A = \{-1, 0, 1, 2\}$ is the *Domain*, $D_f = \{-1, 0, 1, 2\}$ and the set $B = \{0, 1, 2\}$ is known as the *Range*, $R_f = \{0, 1, 2\}$. Note that the **range** of *f* is the set of all possible values of *f(x)* as *x* varies throughout the domain, that is,

$$\text{range of } f = \left\{ f(x) \,\middle|\, x \in A \right\}.$$

If x A, then $f(x)$ [B] is the *image* of x under *f*. The subset of *B* consisting of those elements that are images of elements of *A* under *f* that is the set {*y*: *y* *f*(*x*), for some *x* in *A*} is the *range* of the function (R_f). This means that it is possible not to have all the elements in set B being mapped onto a value in the domain. In this case, the range becomes a part of set B. For example, given the set A= $\{1, 2, 3, 4\}$ and B= $\{a, b, c\}$, we can have the following scenarios:

i. $\{(1,a),(2,b),(3,c),(4,c)\}$ and ii) $\{(1,a),(2,b),(3,a),(4,b)\}$.

You would have noticed that in i), all the elements in set B are paired with an element in set A. However, in the case of ii) the element '*c*' is not the paired with any element in set A. In the case of ii), the range becomes a part of set B.

In summary, you can think of the domain as containing the values of *x* to be substituted into a given formula or rule with the range being the answers obtained after the simplification. Let us take an example to explain this point.

Worked Example 62

Given that $f(x) = 3x - 1$, find the values of $f(x)$ for x $x = -1, 0, 1, 2.$

When $x = -1;\quad f(-1) = 3(-1) - 1 = -4$

When $x = 0;\quad f(0) = 3(0) - 1 = -1$

When $x = 1;\quad f(1) = 3(1) - 1 = 2$

When $x = 2;\quad f(2) = 3(2) - 1 = 5$

In this example, the domain, $D_f = \{-1, 0, 1, 2\}$ while the range, $R_f = \{-4, -1, 2, 5\}$.

Worked Example 63

In this example, we shall determine the domain and range of a function represented in set notation form. Find the domain of the function: $\{(a, 2), (b, 3), (c, 4)\}$. We hope you got the following: domain, $D_f = \{a, b, c\}$ and the range, $R_f = \{2, 3, 4\}$?

Now, your turn!

1. Find $f(-2), f(0), and\ f(6)$ for $f(x) = \sqrt{x+3}.$
2. Evaluate $f(x) = \dfrac{1}{1-x}$ at $x = 2\ and\ x = \dfrac{1}{3}.$

Hopefully, you got the following answers for question 1:

$f(-2) = 1$; $f(0) = \sqrt{3}$ and $f(6) = 3$? And for question 2, you got the answers to be: $f(2) = -1$ and $f(\frac{1}{3}) = \frac{3}{2}$. That is good, keep it up!

So far, we have looked at how to write the domain and range of a function in a set form. It must be noted that every function has a domain however, when the function is described in the form of an equation, there is a need to determine the domain (input or independent variable) of the given function. For some functions, e.g. $f(x) = 3x - 2,$ every input value from the set of real numbers will have its corresponding image. In such cases, it is not possible to list all the elements in the domain because we have an infinite number of elements. It is therefore helpful to write the elements in the domain using: 1) set builder notation and 2) **interval notation**. These two approaches can also be used in writing the elements of the range of a given function if there an infinite number of sets.

Interval notation form involves the use of brackets enclosing values which hold true for a given function. There are different types of brackets that are used depending on whether or not the endpoint of the interval is included or not. This is commonly applied when we have the function in the form of an equation or rule requiring that we determine the domain and range.

Conventions of Interval Notation

1. Write the smallest number in the interval first.
2. Write the largest number in the interval second with a comma separating the two numbers.
3. Use brackets or parenthesis to enclose the two numbers also referred to as the endpoint values.

i. Parenthesis, ( ), are used when the endpoint value is **not** part of the elements in the domain, called **exclusive**.

ii. Brackets, [ ], are used when the endpoint value is part of the elements in the domain, called **inclusive**.

Inequality	Description	Interval notation
$x > a$	*x* is greater than *a*	(a, ∞)
$x < a$	*x* is less than *a*	$(-\infty, a)$
$x \geq a$	*x* is greater than or equal to *a*	$[a, \infty)$
$x \leq a$	*x* is less than or equal to *a*	$(-\infty, a]$
$a < x < b$	*x* is strictly between *a* and *b or x* is greater than ***a*** but less than ***b***	(a, b)
$a \leq x < b$	*x* is greater than or equal to ***a*** but less than ***b***	$[a, b)$
$a < x \leq b$	*x* is greater than ***a*** but less than or equal to ***b***	$(a, b]$
$a \leq x \leq b$	*x* is greater than or equal to ***a*** but less than or equal to ***b***	$[a, b]$

Finding the Domain of a Function

In determining the domain and range of functions, there are three possibilities:

1) If equation of the function has **no** denominator or an even root (since even roots are defined for only nonnegative numbers). In this case, you consider whether the set of all real numbers can be the domain.

2) If the equation of the function has a denominator. In this case, you have to determine the values of *x* in the denominator which makes it zero. You do this by equating the denominator to zero and solving for *x*. After solving the resulting equation, the domain will **not** include the solution(s) to this equation.

3) If there is an even root, set the quantity under the sign greater than or equal to zero to find the domain. That is, if there is an even square root, consider excluding values that would make the radicand negative.

Now, let us solve some examples.

Worked Example 64

Find the domain of the function: $f(x) = x^2$. In this example, since any real number can be squared, the domain for this given function is the set of all real numbers *x* for which $f(x)$ is a real number. That is, domain, $D_f = \{x : x \in R, x \text{ is real}\}$ or $(-\infty, \infty)$.

Worked Example 65

Find the domain of the function: $f(x) = \frac{1}{3}x - 1.$ In this example, the given function has no denominator just as in Example 1. As a result, the domain of the function is the set of all real numbers *x* for which $f(x)$ is a real number. That is, domain, $D_f = \{x : x \in R, x \text{ is real}\}$ or $(-\infty, \infty)$.

Worked Example 66

Find the domain of the function: $f(x) = \frac{x-2}{x-3}.$ In this example, you would notice that we have a rational function; a denominator which is $x-3.$ If the denominator equals zero, we have an undefined function. Try dividing any number by zero using your calculator. What result did you obtain, syntax error? Because of this, we cannot let the expression $x - 3 = 0.$ We therefore solve for the value of x which will make the expression zero, that is, $x = 3.$

The domain of the rational function is therefore the set of all real numbers *x* for which $f(x)$ is a real number, except $x = 3.$ That is, domain, $D_f = \{x : x \in R, x \text{ is real}, x \neq 3\}.$

Worked Example 67

Find the domain of the function, $f(x) = \frac{x-2}{x^2-4}$.

In this example, we have a rational function, hence a need to ensure that the denominator is not equal to zero. You would have noticed that the denominator can be expressed as a difference of two squares, $x^2 - 4 = (x-2)(x+2).$

In that case, $f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}$.

Setting the denominator to zero, we have $x + 2 = 0$, $x = -2.$ The domain of the rational function is therefore the set of all real numbers *x* for which $f(x)$ is a real number, except $x = -2.$ That is, domain, $D_f = \{x : x \in R, x \text{ is real}, x \neq -2\}.$

This example also shows that sometimes, we can simplify the given expression before proceeding with any other computations.

Worked Example 68

Find the domain of the function, $f(x)=\sqrt{9-t}$.

In functions involving square roots (radicals), we exclude any real number(s) that will give a negative number under the radicand (number under the square root).

Therefore, we set the radicand to be greater than or equal to zero and solve for the variable.

$$9-t\geq 0$$

$$9\geq t \text{ or } t\leq 9.$$

What this means is than any value of *t* greater than 9 will result in a negative number under the radicand and so should be excluded from the domain. Alternatively, we are interested in the values of *t* which are less than or equal to 7 as elements in the domain.

The domain of the given function is therefore the set of all real numbers less than or equal to 9 or $(-\infty, 9]$.

Key ideas

- The set of input values is the **domain** and the set of output values is called the **range**.
- The **range** of *f* is the set of all possible values of *f(x)* as *x* varies throughout the domain, that is, $\text{range of } f=\{f(x)\mid x\in A\}$.
- The elements in the domain may be **listed** or described using: words, set builder notation, and **interval notation**.

Reflection

1. Given the following function: $\{(2,0),(3,10),(4,20),(5,30),(6,40)\}$, find the:
   i. domain and
   i. range.
2. Given, $f(x)=-2x^2-2x+1$, evaluate the following:
   i. $f(-1)$ *and*
   i. $f(2)$
3. Given, $f(x)=\dfrac{1}{x^2+1}$, evaluate the following:
   i. $f(-1)$ *and*
   i. $f(\frac{1}{2})$
4. Evaluate: $f(-b)$ for $f(x)=3x^2-2x+3$
5. Find the domain of the following functions:
   i. $f(x)=\dfrac{x^2-9}{x+3}$.
   i. $f(x)=\dfrac{1}{x^2-6x+9}$.
   i. $f(x)=\dfrac{x^2+x-5}{\sqrt{3-x}}$.

SESSION 3: ALGEBRA OF FUNCTIONS

In this session, we will consider introducing participants to performing the four basic operations (addition, multiply, subtraction, and division) on functions.

Learning outcomes

By the end of the session, the participant will be able to:

1. add, multiply, subtract and divide functions;

Suppose that you are a salaried worker who receives a monthly income of *x*. We can write this as a function of the form say, $m(x).$ If you should earn from extra teaching at the end of each month, we can also write the amount earned as a function, $e(x).$ In this case, your total income for each month is obtained by adding your earnings from the two sources as follows:

$$T(x) = m(x) + e(x).$$

What this means is that we can add given functions just as we can add real numbers. In fact, any of the four basic operations we perform on the set of real numbers can also be performed on functions, namely addition, subtraction (difference), multiplication (product) and division. When we combine two or more functions using any of the operation(s), we obtain a 'new' function.

Given the functions, $f(x)$ and $g(x)$ with real number values, we can perform algebraic operations on them to obtain new functions as follows:

a) $f(x) + g(x) = (f+g)(x)$
b) $f(x) - g(x) = (f-g)(x)$
c) $f(x)\ g(x) = (fg)(x)$
d) $\dfrac{f(x)}{g(x)} = (\dfrac{f}{g})(x)$ where $g(x) \neq 0$.

Worked Example 69

Given the functions, $f(x) = x - 1$ and $g(x) = x^2 - 1$, perform the following algebraic operations:

a. $g(x) - f(x)$
b. $f(x) - g(x)$
c) $\dfrac{g(x)}{f(x)}$

Solution

a) $g(x) - f(x) = (g - f)(x)$

$$= (x^2 - 1) - (x - 1)$$
$$= x^2 - 1 - x + 1$$
$$= x^2 - x = x(x-1)$$

b) $f(x) - g(x) = (f - g)(x)$

$$= (x - 1) - (x^2 - 1)$$

$$= x - 1 - x^2 + 1$$
$$= x - x^2 = x(1-x).$$

c) $\frac{g(x)}{f(x)} = (\frac{g}{f})(x)$

$$= \frac{x^2 - 1}{x - 1}, where\ x - 1 \neq 0$$
$$= \frac{(x-1)(x+1)}{x-1}$$
$$\frac{g(x)}{f(x)} = x + 1$$

We hope that you noticed that in all the three cases, the functions obtained are different from what we started with. Again, you may have noticed that $g(x) - f(x) \neq f(x) - g(x).$ What does this suggest about subtraction of functions? Let us take any example to verify whether your deduction is correct.

Worked Example 70

Given that $f(x) = x^2 - 2x + 5$ and $g(x) = 3x - 5$; simplify the following and find their domains in each case.

a) $f(x) + g(x)$
b) $g(x) + f(x)$
c) $f(x) - g(x)$
d) $g(x) - f(x)$
e) $f(x)g(x)$
f) $\frac{f(x)}{g(x)}$

Solutions

a) $f(x) + g(x) = (x^2 - 2x + 5) + (3x - 5) = x^2 + x = x(1 + x)$. The domain is the set of all real numbers or $(-\infty, \infty)$.

b) $g(x) + f(x) = (3x - 5) + (x^2 - 2x + 5) = x + x^2 = x(x + 1)$. The domain is the set of all real numbers or $(-\infty, \infty)$.

c) $f(x) - g(x) = (x^2 - 2x + 5) - (3x - 5) = x^2 - 5x + 10$. The domain is the set of all real numbers or $(-\infty, \infty)$.

d) $g(x) - f(x) = (3x - 5) - (x^2 - 2x + 5) = 5x - x^2 - 10.$ The domain is the set of all real numbers or $(-\infty, \infty)$.

e) $f(x)g(x) = (x^2 - 2x + 5)(3x - 5) = 3x^3 - 5x^2 - 6x^2 + 10x + 15x - 25 = 3x^3 - 11x^2 + 25x - 25.$ The domain is the set of all real numbers or $(-\infty, \infty)$.

f) $\frac{f(x)}{g(x)} = \frac{x^2 - 2x + 5}{3x - 5}$. The domain is the set of all real numbers except $x = \frac{5}{3}$.

What conclusion(s) can you draw about the addition, subtraction, multiplication and division of functions taking the order of operation into consideration?

Key ideas

- $f(x) + g(x) = (f + g)(x)$
- $f(x) - g(x) = (f - g)(x)$
- $f(x)\ g(x) = (fg)(x)$
- $\frac{f(x)}{g(x)} = (\frac{f}{g})(x)$ where $g(x) \neq 0$.
- $g(x) - f(x) \neq f(x) - g(x)$.

Reflection

1. Given that $h(x) = 2x^2 - x$ and $k(x) = x + 4$, simplify the following:
   a) $h(x) + k(x)$
   b) $k(x) + h(x)$
   c) $h(x) - k(x)$
   d) $k(x) - h(x)$
   e) $h(x)k(x)$
   f) $\frac{h(x)}{k(x)}$
2. Find the domain for each of the algebraic functions in 1) above.
3. Given that $f(x) = x^2$ and $g(x) = x - 3$, simplify $f(x).g(x)$.
4. Given that $f(x) = 3x^2$ and $g(x) = \sqrt{x - 5}$, simplify:
   i) $f(x) + g(x)$
   ii) $f(x) - g(x)$
5. Given that $f(x) = \frac{1}{x - 4}$ and $g(x) = \frac{1}{5 - x}$, simplify:
   i) $f(x) + g(x)$
   ii) $f(x) + g(x)$
   iii) $\frac{f(x)}{g(x)}$

SESSION 4: COMPOSITION OF TWO FUNCTIONS

In this session, we will consider introducing participants to how to determine the composite of given function.

Learning outcomes

By the end of the session, the participant will be able to:

1. determine the composite of given functions;

Let us take an example to explain this. Given that Ama earns Gh₵ 220.00 a week as a sale agent and an additional 3% commission when she records sales over Gh₵ 5,000.00. How much will she take home at the end a week that she records Gh₵ 7,000.00?

In this example, we can formulate two functions for her commission; the first is that she is to get 3% for exceeding an amount of Gh₵ 5,000.00 , that is, $f(x) = 0.03x$ and the actual amount that attracts the commission, $g(x) = x - 5000.$ This gives an example of a composite function because to calculate the commission Ama will earn, we first determine the amounts that attracts the commission, $g(x) = x - 5000 = 7000 - 5000 = Gh \not\subset 2000.00$. Following this, we can determine the actual commission for Ama: $f(x) = 0.03x = 0.03 \times 2000 = Gh \not\subset 60.00$.

From this example, you would observe that the result (**output)** from the function, $g(x)$ which is $Gh \not\subset 2000.00$ became the **input** for $f(x).$ We can represent it in function notation as $(fog)(x) = f(g(x))$.

What we just formed illustrates the concept of **composition**. In composition, what we do in forming a new function is that the **output** of a first function, in our example, $g(x)$ becomes the **input** of a second function, $f(x).$

Your turn: Assuming that you bought 200 bags of cement from a hardware dealer who also charges a small amount for out-of-town deliver. As such, you have to pay for the following: cost of 200 bags of cement, 17.5 percent VAT and Gh₵ 50.00 for out-of-town delivery.

a. Write a function for the amount you should pay for the bags of cement (quantity and VAT), excluding the delivery fee.

[Ans. $f(x) = 0.175x \text{ where } x \text{ represents number of cement bags purchaced}$ **]**

b. Write a function for the amount you should pay for the bags of cement (quantity) including the delivery fee. [Ans. $g(x) = 20.$]

c. Write a composite function for the given information. Ans. $(fog)(x) = f(g(x)).$

We read the function $(fog)(x) = f(g(x))$ as follows: the left-hand side is "*f* composed with *g* at *x*" and the right-hand side as "*f* of *g* of *x*". Since there is an equation connecting the two sides, they are equal in value. The symbol "*o*" is called the **composition operator**.

Additionally, the order in which we operate on composite functions is important since in most cases $(f(g(x)) \neq g(f(x)).$ With the left-hand side, $f(x)$ takes the output of $g(x)$ as its input while on the right-hand side, $g(x)$ now takes the output of $f(x)$ to be its input. **Try with the Ama's example and see if you will get the same results? I hope not.**

Let us solve some examples.

Worked Example 71

Given that $f(x)=x^2$ and $g(x)=x-2$, evaluate:

i. $(fog)(x)$.

ii. $(gof)(x)$.

iii. What conclusion can you draw from your answers in i) and ii)?

Solutions

i. $(fog)(x)=f(g(x))=(x-2)^2=x^2-4x+4.$

ii. $(gof)(x)=g(f(x))=(x)^2+2=x^2+2.$

iii. $(fog)(x)\neq(gof)(x).$ That is, the composition of the two functions is **not** commutative.

Worked Example 72

Given that $h(x)=x^2-1$ and $k(x)=3x-2$, evaluate:

i. $(hok)(x)$.

ii. $(koh)(x)$.

iii. What conclusion can you draw from your answers in i) and ii)?

Solutions

i. $(hok)(x)=h(k(x))=(3x-2)^2-1=9x^2-12x+4-1=9x^2.-12x+3$

ii. $(koh)(x)=k(h(x))=3(x^2-1)-2=3x^2-3-2=3x^2.-5$

iii. $(hok)(x)\neq(koh)(x).$ That is, the composition of the two functions is **not** commutative.

Worked Example 73

Given that $f(x)=\sqrt{3x-2}$ and $g(x)=x^2$, evaluate:

i. $(fog)(x)$.

ii. $(gof)(x)$.

iii. What conclusion can you draw from your answers in i) and ii)?

Solutions

i. $(fog)(x)=f(g(x))=\sqrt{3x^2-2}.$

ii. $(gof)(x)=g(f(x))=(\sqrt{3x-2})^2=(\sqrt{3x-2})^2$

iii. $(fog)(x)\neq(gof)(x).$ That is, the composition of the two functions is **not** commutative.

Worked Example 74

Given that $f(x)=\dfrac{2}{1+x}$ where $x\neq1$ and $g(x)=\dfrac{3x+1}{x-3}$, where $x\neq3$ evaluate:

i. $(fog)(x)$.

ii. $(gof)(x)$.

Solutions

i. $$(fog)(x) = f(g(x)) = \frac{2}{1+\frac{3x+1}{x-3}} = \frac{x-3}{2x-1}.$$

ii. $$(gof)(x) = g(f(x)) = \frac{3(\frac{2}{1+x})+1}{(\frac{2}{1+x})-3} = \frac{7+x}{-1-3x}$$

In the next two examples, we shall look at how to evaluate the composite function when *x* is assigned a particular value.

Worked Example 75

Given that $f(x) = x^2$ and $g(x) = x-2$, evaluate:

i. $fog(1)$.

ii. $fog(-1)$.

iii. $gof(0)$.

Solutions

i. $fog(1) = f(g(1)$. That is, we first evaluate *x*=1 using $g(x), i.e. g(1)$

$$g(1) = 1-2 = -1.$$

Now, we evaluate $f(x)\ at\ x = -1$

$$f(-1) = (-1)^2 = 1$$

ii. $fog(-1)$.

We begin by evaluating $g(x)\ at\ x = -1$

$$g(-1) = -1-2 = -3.$$

Now, we evaluate $f(x)\ at\ x = -3$

$$f(-3) = (-3)^2 = 9$$

iii) $gof(0)$. We begin by evaluating $f(x)\ at\ x = 0$

$$f(0) = (0)^2 = 0$$

Now, we evaluate $g(x)\ at\ x = 0$

$$g(0) = 0-2 = -2$$

Worked Example 76

Given that $f(x) = x^2 - 3x$ and $g(x) = x-2$, evaluate:

i. $fog(1)$.

ii. $gof(-5)$.

Solutions

i. $fog(1)$. We start by evaluating $g(x)$ at $x = 1$.

$$g(1) = 1-2 = -1.$$

We now evaluate $f(x)$ at $x = -1$.

$$f(-1) = (-1)^2 - 3(-1) = 4.$$

ii) $gof(-5)$. We begin with the $f(x)$ at $x=-5$.

$f(-5)=(-5)^2-3(-5)=40$.

We now evaluate $g(x)$ at $x=40$

$g(40)=40-2=38.$

Key ideas

- $(fog)(x)=f(g(x))$
- The symbol "*o*" is called the **composition operator**.
- $(f(g(x))\neq g(f(x)).$

Reflection

1. Given $f(x)=x^3+1$ and $g(x)=\sqrt[3]{x-1}$, evaluate:

i. $(fog)(x)$.

i. $(gof)(x)$.

i. What conclusion can you draw from your answers in i) and ii)?

2. Given $f(x)=3x^2+x$ and $g(x)=x+7$, evaluate:

i. $(fog)(x)$.

i. $(gof)(x)$.

3. Given $f(x)=\dfrac{1-x}{x}$ and $g(x)=\dfrac{1}{1+x^2}$, evaluate :

i. $(fog)(x)$.

i. $(gof)(x)$.

iii) $gof(-3)$.

4. For the following questions, use the function tables for *f* and *g* shown in the table below to evaluate each expression.

x	0	1	2	3	4	5	6	7	8	9
f(x)	7	6	5	8	4	0	2	1	9	3
g(x)	9	5	6	2	1	8	7	3	4	0

a) $fog(2)$. b) $fof(4)$. c) $gof(5)$.

SESSION 5: INVERSES OF FUNCTIONS

Remember when you were learning about how to operate on the set of real numbers. You learned about the identity elements for addition and multiplication as zero (0) and one (1) respectively. Also, you learned how to find the additive and multiplicative elements for given numbers. When it comes to functions, we can also determine the inverse of a given function, provided the inverse exists. The notation for inverse function is $f^{-1}(x)$ read as "*f* inverse of *x*" Note that this notation is **not** the same as $(f(x))^{-1}$ which represents the multiplicative inverse of $f(x)$.

Learning outcomes

By the end of the session, the participant will be able to:

- find the inverse of simple functions;

Determining Inverse of Two Given Functions

Given a function $f(x)$, we can determine whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether $(fog)(x) = x$ or $(gof)(x) = x$ holds.

For example, given $f(x) = 7x$ and $g(x) = \frac{1}{7}x$, we can have $(gof)(x) = \frac{1}{7}(7x) = x$ *or* $(fog)(x) = 7(\frac{1}{7}x) = x$. Since the composite of the two functions gives x, the two functions are inverses functions.

In effect, given two functions $f(x)$ and $g(x)$ we can determine whether they are inverses of each other by:

1. Verifying whether $(fog)(x) = x$ or $(gof)(x) = x$. That is, check whether the left hand-side or the right hand-side is true. It is not necessary to check for both statements.
2. If either statement is true, then both are true implying $f(x) = g^{-1}(x)$ and $g(x) = f^{-1}(x)$.
3. If either statement is false, then both are false implying $f(x) \neq g^{-1}(x)$ and $g(x) \neq f^{-1}(x)$. .

Let us take an example.

Worked Example 77

Determine whether $f(x) = 2x + 5$ and $g(x) = \frac{1}{2}x - \frac{5}{2}$ are inverses.

To do so, we can check whether $(fog)(x) = x$ or $(gof)(x) = x$.

$$(fog)(x) = f\ (\frac{1}{2}x - \frac{5}{2})$$

$= 2(\frac{1}{2}x - \frac{5}{2}) + 5 = x - 5 + 5 = x$ as required.

Although it is not necessary to evaluate for $(gof)(x) = x$, we shall to that for confirmation purposes.

That is, $(gof)(x) = g(2x+5)$

$$= \frac{1}{2}(2x+5) - \frac{5}{2}$$

$$= x + \frac{5}{2} - \frac{5}{2} = x$$

Since either $(fog)(x) = x$ or $(gof)(x) = x$, it means that $f(x)$ and $g(x)$ are inverses of each other.

Worked Example 78

Determine whether $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1}{x} - 2$ are inverses, that is, is $g(x) = f^{-1}(x)$?

$(fog)(x) = f(g(x)) = \dfrac{1}{(\frac{1}{x} - 2) + 2} = \dfrac{1}{\frac{1}{x}} = x$. Therefore, $g(x) = f^{-1}(x)$ as required.

Worked Example 79

Determine whether $f(x) = x^3$ and $g(x) = \dfrac{1}{3}x$ are inverses, that is, is $g(x) = f^{-1}(x)$?

For the two given functions to be inverses of each other, we expect that either $(fog)(x) = x$ or $(gof)(x) = x$

$(fog)(x) = f(g(x)) = (\frac{1}{3}x)^3 = \frac{1}{27}x^3 \neq x$.

Since $(fog)(x) \neq x$, the two functions are **not** inverses of each other.

In the previous examples, we determined the inverse of two given functions by determining whether their composition equals *x* or not. We shall now turn our attention to how to determine the inverse of a function (only one function).

Determining Inverse of a Given Function

If the given function, in terms of *x*, is represented in the form of an equation (rule/formula), we can determine the inverse by:

1. interchanging *x* and *y* in the given equation and;
2. Solving to make *x* the subject, i.e. make *x* a function of *y*.

Worked Example 80

Determine the inverse of $f(x) = \dfrac{1}{3}(x-5)$.

Given $f(x) = \frac{1}{3}(x-5)$, we let $y = f(x)$

$\Rightarrow y = \frac{1}{3}(x-5)$. That is, we obtain an equation.

$3y = x - 5$. Multiplying through by 3

$3y + 5 = x - 5 + 5$. Making *x* the subject

$\therefore x = 3y + 5$. That is, $f^{-1}(y) = 3y + 5$ or $f^{-1}(x) = 3x + 5$.

Worked Example 81

Determine the inverse of the function $f(x) = \frac{1}{x-4} + 3$.

Let $y = f(x)$

$$\Rightarrow y = \frac{1}{x-4} + 3$$

$$y - 3 = \frac{1}{x-4}$$

$$x - 4 = \frac{1}{y-3}$$

$\therefore x = \frac{1}{y-3} + 4$. That is, $f^{-1}(y) = \frac{1}{y-3} + 4$ or $f^{-1}(x) = \frac{1}{x-3} + 4$.

Key ideas

- $f^{-1}(x)$ read as "*f* inverse of *x*" is **not** the same as $(f(x))^{-1}$ which represents the multiplicative inverse of $f(x)$.
- Given a function $f(x)$, we can determine whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether $(fog)(x) = x$ or $(gof)(x) = x$ holds.

Reflection

Find the inverse functions of the following:

1. $f(x) = x^3 + 1$

2) $g(x) = \frac{x-5}{-3}$

3) $f(x) = \frac{2x+3}{5x+4}$

4) $f(x) = 2 - \sqrt{x}$
5) $h(x) = 2 + \sqrt{x-4}$

SESSION 6: GRAPHS OF FUNCTIONS

In an earlier session, we indicated that functions can be represented in several ways. In this session, we shall focus on one of the ways by which functions can be represented which is, graphs. A visual representation of a function is very important because it reveals a lot of information about a given function. For instance, from a given graph, we can determine: 1) whether it represents a function, 2) the domain and range of the function, where the function is increasing or decreasing.

In order to construct a graph, we need some input (*x*-values/independent variable) and output (*y*-values/dependent variable). The graph of the function $y = f(x)$ represents the set of all points (x, y) in the Cartesian plane that satisfies the equation $y = f(x)$.

Learning outcomes

By the end of the session, the participant will be able to:

- graph any given function.

Reading Input and Output Values from a Given Graph

Let us use the graph of $y = x^2$ (Figure 1). From the graph, $f(3) = 9$; $f(-1) = 1$ and $f(0) = 0$. You would notice that these are not the only points lying on the curve. There are more such as: $(-3, 9)$ and $(-2, 4)$ which can also be written as $f(-3) = 9$ and $f(-2) = 4$ respectively. Can you find three more points (x, y) that lie on the given graph?

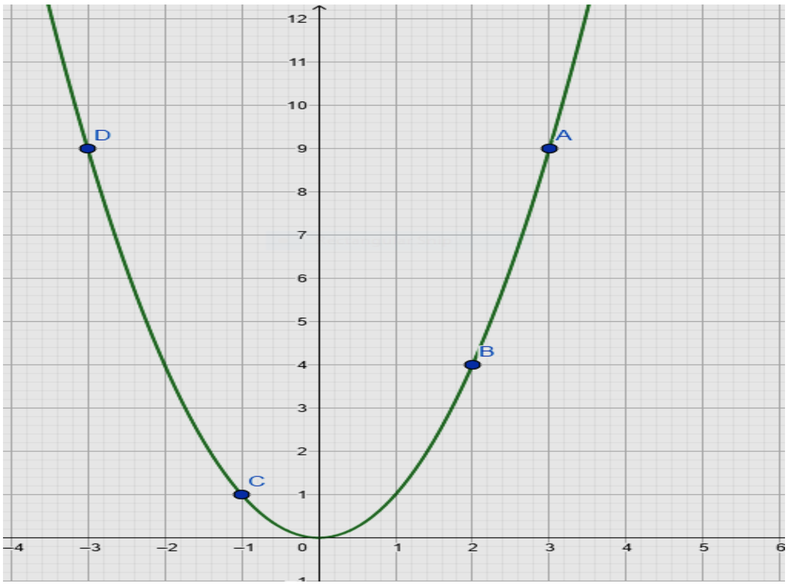


Figure 2.1

Let us examine another graph in Figure 2.1. For the three dots on the graph, we have the following coordinates: $A(1, 1)$; $B(-1, -1)$ *and* $C(-2, -8)$.

Determine as many points as possible that lie on the graph shown in Figure 2.

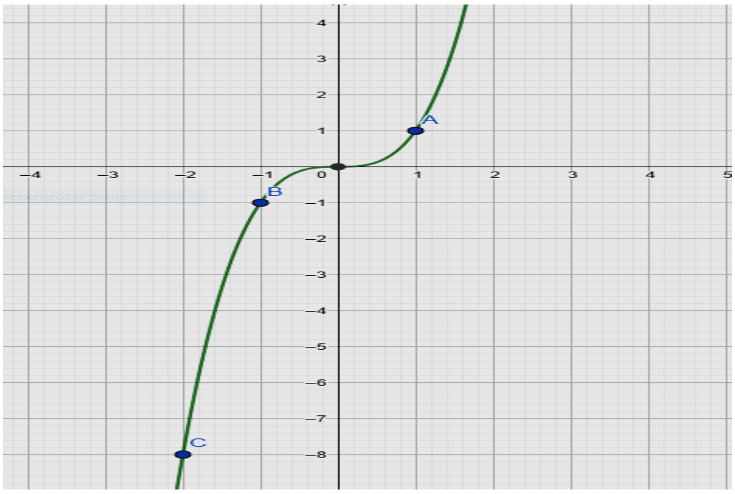


Figure 2.2

From the two examples, we can observe that from any given graph, we can form a table of values being identifying some points that lie on the graph and writing it in coordinate form. Also, given a table of values you can draw an appropriate graph for it.

Determining whether a Graph Represents a Function

Although we draw various graphs, not all of them represent the graph of a function. The question then is, how can we determine whether a graph represents a function? We can do so by performing a **vertical line test.** On the given graph, we draw a vertical line to cut through it. If the vertical line intersects a graph at only once, then the graph defines a function. This is because we define a function as each input value having one and only one output value. That is, if any vertical intersects a graph more than once, then the graph is **not** a function.

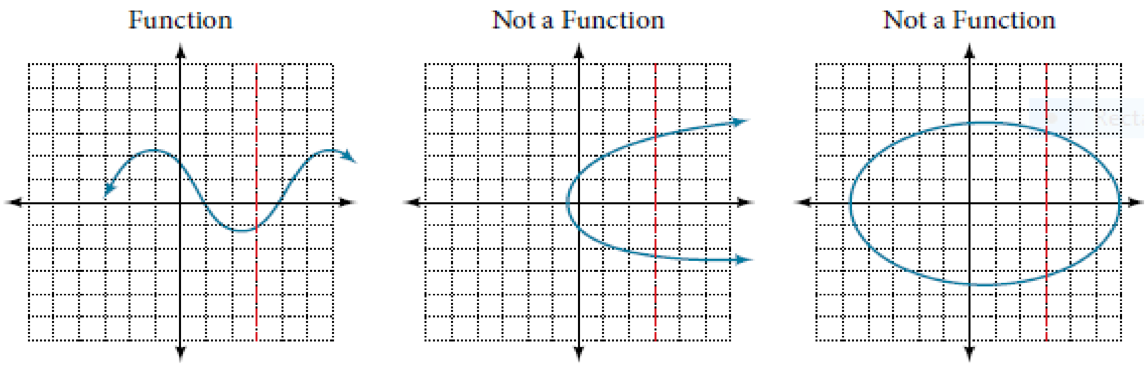


Figure 2.3

Now, your turn! Determine which of the following graphs represent functions?

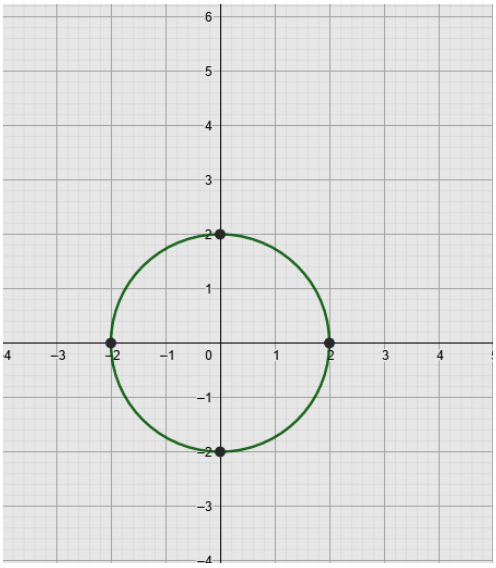


Figure 2.4

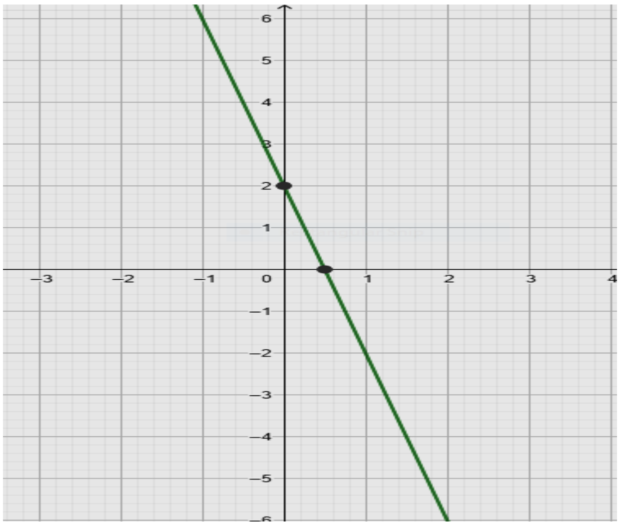


Figure 2.5

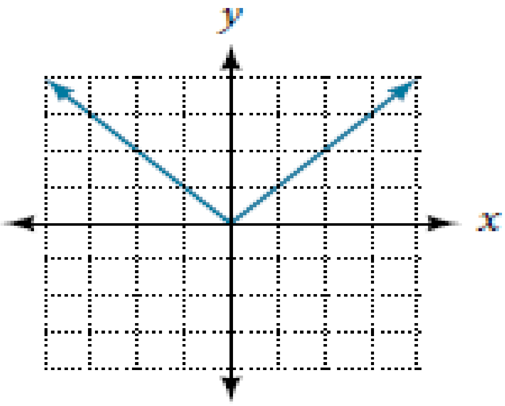


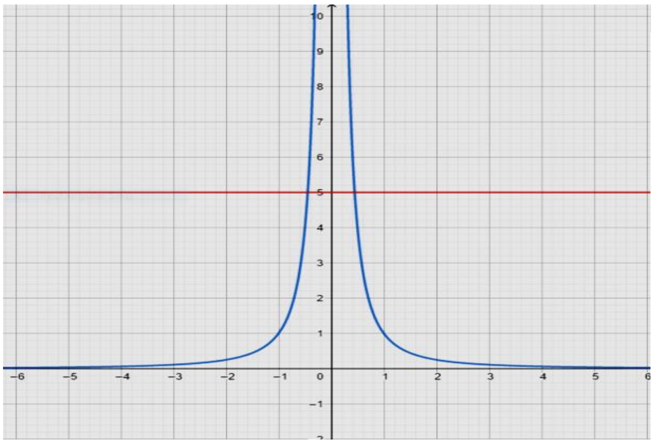
Figure 2.6

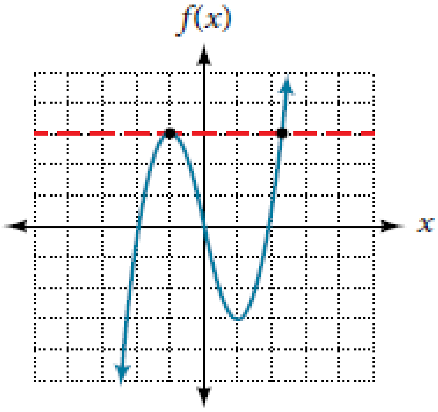
If you had Figures 2.5 and 2.6 to be functions, using the vertical line test, well done!

Determining whether a function is One-to-One

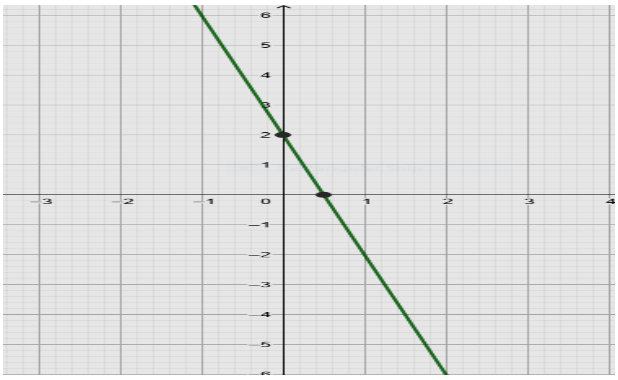
As indicated earlier, a graph reveals a lot of information. So far, we have learned how to obtain the input and value values from a given graph and also how to determine whether the graph represents a function or not. Now, we shall turn our attention to information that can be obtained from a graph which is, o**ne-to-one function**.

Once it has been determined that a graph defines a function, we can perform a **horizontal line test.** To do this test, draw a horizontal line through the graph. If this horizontal line intersects the graph more than once, then the graph **does not** represent a one-to-one function. For example, the following graphs although represent functions are **not** one-to-one functions because the horizontal line intersects the graphs at two places.





However, the following graph represents a one-to-one function because any vertical line drawn will intersect the graph only once. Try to see whether it is true or not.

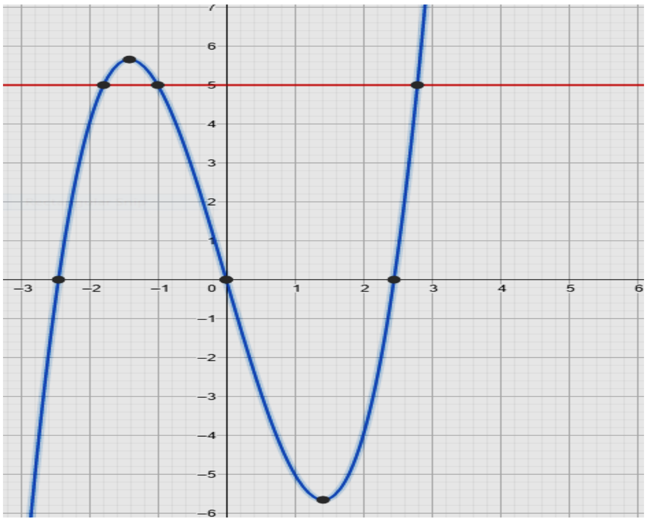


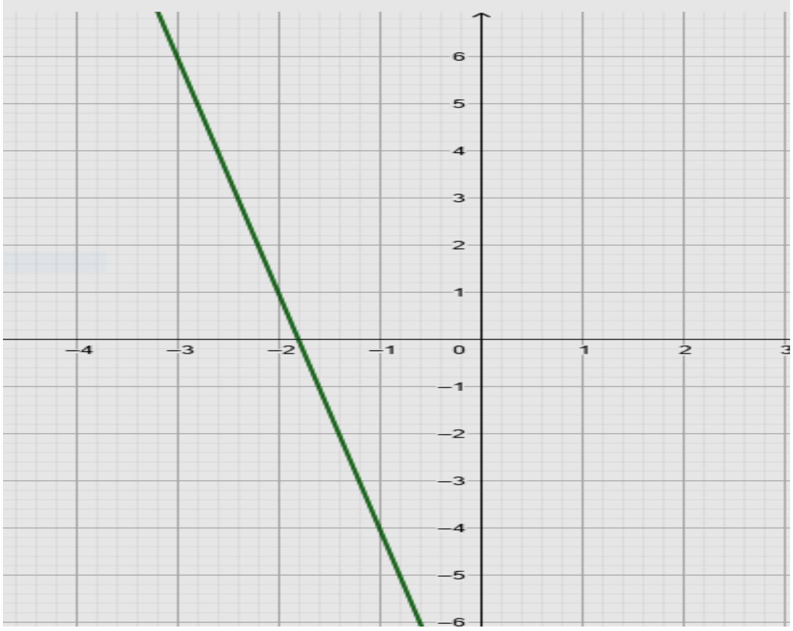
Key ideas

- The graph of the function $y = f(x)$ represents the set of all points (x, y) in the Cartesian plane that satisfies the equation $y = f(x)$.
- We can determine whether a graph represents a function by performing a **vertical line test.**
- If the vertical line intersects a graph at only once, then the graph defines a function.
- When also determine whether a function is o**ne-to-one** by performing the **horizontal line test.**
- If this horizontal line intersects the graph more than once, then the graph **does not** represent a one-to-one function.

Reflection

Determine whether the following graphs represent functions or not. For those that are functions, identify those that are one-to-one functions.





6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−4
−3
−2
−1
1
2
3

UNIT 7: ALGEBRAIC EXPRESSIONS AND EQUATIONS (LINEAR EQUATIONS AND QUADRATICS)

In this unit, we introduce participants to algebraic expressions and equations. We will use algebraic tiles to guide participants to factorize algebraic expressions (linear and quadratic) and to expand products of two binomials up to the form $(ax + b)(x + a)$. We will also expose participants to various problem-solving strategies to equip them to tackle word problems.

Learning outcome(s)

By the end of the unit, the participant will be able to:

1. factorise algebraic expressions up to four terms
2. solve simple linear equations
3. graph linear and quadratic equations
4. solve word problems involving simple linear equations

SESSION 1: ALGEBRAIC EXPRESSIONS

In your study of mathematics, you have become used to operating on numbers such as $7-15+3-18$. It is not always that you have to simplify numeric expressions. In Algebra, we make use of both numbers and variables (x, y, z, etc.). Algebraic expressions are made up of terms where a **term** is made up of a constant (number) or the product of a constant and one or more variables. A **variable** is any symbol which can be replaced by any member of a set of numbers. The terms which make an algebraic expression can be combined using any combinations of the four basic operations namely; addition, multiplication, subtraction and division. Examples of a terms are: $7, x, 5b^2, 8a \ and \ y^7$. The constant term, e.g. 5, which multiples the variable, b, is called the **coefficient**. In lay terms, the coefficient is the number in front to the variable. For example, the coefficient of the term $5t$ is 5 while the coefficient of the term y is 1 since $1\times y = y$.

TRY

Identify the coefficient of each of the following terms:

a) $-17t$ b) $-a$ c) a d) $7m$ e) $-\frac{1}{4}n$ f) $\frac{5}{4}t$

Like Terms

Consider the following terms and determine which of them have common characteristics:

a) $5y \quad 9 \quad m^3 \quad 5 \quad 2y \quad 8m^3$

I hope that you made the following observations:

i. The 9 and 5 are both constant terms,

ii) The 5y and 2y are both terms with y,

iii) The $m^3 \ and \ 8m^3$ are both terms with m^3.

When two terms are constants or have the same variable and exponent, we say they are **like terms.** Therefore, 9 and 5 are like terms; 5y and 2y are like terms; and $m^3 \ and \ 8m^3$ are like terms.

TRY

Identify the like terms in the following: $x^3,\ 18x^2, 11x^3, 7x, -3x, 9x^2, 2t, 4r^2$

As indicated earlier, when we perform any of the basic operations on any number of terms, we obtain an algebraic expression. For example, we can think of the expression; $x^2 + 7x + 5$ as combining three terms which are: $x^2, 7x, and\ 5.$

Addition and Subtraction of Algebraic Expressions

Just as we can simplify numeric expressions, so can we simplify algebraic expressions provided there are like terms in the given expressions. That is, if there are like terms in a given expression(s), you can simplify the expression(s) by combining the like terms. For example, you can simplify $13x + 7x + 5x$ to __________? If you got an answer of $25x$ then you are right!

Hopefully, you got the answer by adding the coefficients; 13, 7 and 5, and keeping the same variable. This is similar to having 13 books then you add 7 more of the books and then later 5 more of the same books. You'll end up with 25books.

So, how do we simplify any given algebraic expressions?

1. Identify the like terms.
2. Rearrange the expression so the like terms are grouped together.
3. Combine the like terms using the given operations.

Let's take some examples and simplify using our procedure.

a. Find the sum of $3x^2 + 7x + 9$ and $5x^2 + 7x + 6$

b. Subtract $(6y^2 - 5y - 2)$ from $(4y^2 - 2y + 7)$

a.

Step	**Action Taken**
1. Identify the like terms.	Given: $3x^2 + 7x + 9$ and $5x^2 + 4x + 6$ $3x^2 + 7x + 9 + 5x^2 + 4x + 6$ The like terms are $3x^2\ \&\ 5x^2$; $7x\ \&\ 4x$ and 9 &6
2. Rearrange the expression so the like terms are grouped together	$3x^2 + 5x^2 + 7x + 4x + 9 + 6$
3. Combine the like terms using the given operations.	$8x^2 + 11x + 15$

Step	Action Taken
1. Identify the like terms.	Given: Subtract $(6y^2 - 5y - 2)$ from $(4y^2 - 2y + 7)$ $(4y^2 - 2y + 7) - (6y^2 - 5y - 2) =$ $4y^2 - 2y + 7 - 6y^2 + 5y + 2$ The like terms are $4y^2$ & $-6y^2$; $-2y$ & $5y$ and 7 & 2
2. Rearrange the expression so the like terms are grouped together	$4y^2 - 6y^2 - 2y + 5y + 7 + 2$
3. Combine the like terms using the given operations.	$-2y^2 + 3y + 9$

TRY

Simplify the following:

1. $3y^2 + 7y + 9 + 7y^2 + 9y + 8$
2. $m^4 - 3np + p^2 + 5m^4 - 7np - 2p^2$
3. $4x^2 + 5x - (-2x - 3y) - (-3x - 6y)$

Multiplication of Algebraic Expressions

In the previous sub-session we looked at how to add or subtract algebraic expressions. We now turn our attention to how to multiply two expressions. We take an example to illustrate it. Multiply $(4y - 2y)$ by $(y - 3y)$. In multiplying, we take each term in the first bracket and use it to multiply each term in the second bracket as follows:

$$(4y - 2y)\ (y - 3y) = 4y\,(y - 3y) - 2y(y - 3y)$$

$$= 4y^2 - 12y^2 - 2y^2 + 6y^2$$

$$= -8y^2 + 4y^2 = -4y^2$$

Let's take another example; $= (t + 4)(t - 5)$

$$= t(t - 5) + 4(t - 5)$$

$$= t^2 - 5t + 4t - 20$$

$$= t^2 - t - 20$$

TRY

Simplify the following expressions:

1. $4t + t(13 - 7)$

2. $5m \div 3m(9 - 6)$

3. $6 + 12a - 3(6b)$

4. $9b + 4b(2 + 3) - 4(2b + 3b)$

5. $= \frac{a}{2^3}(64) - 12a \div 6$

6. $4t - 4(t - \frac{5}{4}s) - (\frac{2}{3}t + 2s)$

7. $5(-n - 3) - (n + 2)$

8. $mn - 5m + 3mn + n$

Factorisation of Algebraic Expressions

We have already learned how to combine terms to form algebraic expressions. Just as we can multiply expressions to obtain a new expression, we can also reverse this process through factorisation. We shall take some examples to illustrate the concept of factorisation.

E.g. 2 Factorise $3x + 21y$

Step	**Action taken**
1. Identify the Greatest Common Factor (GCF) of the coefficients	The coefficients of the given expression are 3 & 21. The GCF of these two numbers is 3.
2. Identify the GCF of the variables	The variables are x & y. In this case, there is no common factor so we leave it as such.
3. Combine to find the GCF of the expression	That is, 3.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression	For the term, $3x$, we have (x) and for $21y = 3(7y)$
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.	$3x + 21y = 3(x + 7y)$

E.g. 2. Factorise: $6x^3y^3 + 45x^2y^2 + 21xy$

Step	Action taken
1. Identify the Greatest Common Factor (GCF) of the coefficients	The coefficients of the given expression are 6, 45 & 21 respectively. The GCF of these numbers is 3.
2. Identify the GCF of the variables	The variables are $x^3, y^3, x^2, y^2, x \,\&\, y$ & y. In this case, for variables in terms of x we have the GCF is x while for the y - variables, the GCF is y.
3. Combine to find the GCF of the expression	That is, $3xy$.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression	For the terms we have : $6x^3y^3 = 3xy(2x^2y^{2)}$, $45x^2y^2 = 3xy(15xy)$ and $21xy = 3xy(7)$
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.	$(3xy)(2x^2y^2 + 15xy + 7)$

Hint. For each of the factored expressions in the solved examples, expand them to verify whether you get the original expressions.

TRY

Factor:

1. $x(b^2 - a) + 6(b^2 - a)$
2) $16a^5b^4 - 24a^2b^5 - 8a^3y^3$
3) $x^3y^2 + 8x^2t^2 - x^3 - 8t^2$

Key ideas

- A **variable** is any symbol which can be replaced by any member of a set of numbers.
- Algebraic expressions are made up of terms where a **term** is made up of a constant (number) or the product of a constant and one or more variables.

Reflection

Simplify the following:
• $m^4 - 3np + p^2 + 5m^4 - 7np - 2p^2$
• $4x^2 + 5x - (-2x - 3y) - (-3x - 6y)$
• $6 + 12a - 3(6b)$
• $9b + 4b(2+3) - 4(2b + 3b)$
• $mn - 5m + 3mn + n$
• $16a^5b^4 - 24a^2b^5 - 8a^3y^3$
• $x^3y^2 + 8x^2t^2 - x^3 - 8t^2$

SESSION 2: LINEAR EQUATIONS INVOLVING ONE VARIABLE AND ITS APPLICATIONS

It is our expectation that you are making good progress in learning the concepts treated in the first five sessions of this unit. In the few instances you may have faced some challenges we expect that you got support from either your colleagues or course tutor during the face-to-face sessions. Now, let us turn our attention to another important topic in mathematics which has to do with linear equations. In this session, we shall explain some terms used in linear equations and also solve some examples involving linear equations.

Learning outcomes

By the end of the session, the participant will be able to:

- distinguish among expressions, equations and identities,
- solve problems involving linear equations.

Concepts of Expressions, Equations and Identities

Expressions, equations and identities

$7x - 5$ is an example of an **algebraic expression** whereas $7x - 5 = 2$ is an example of an **equation**. So what is the difference, you may be asking? An equation is simply a statement that two or more quantities are equal. A distinguishing feature of an equation is that it contains an 'equals' sign whereas an expression does not. An **identity**, on the other hand, is a relationship which is true for all values of the unknown. An identity, therefore, is different from an equation despite both having an 'equals' sign. Whereas an equation is true only for some particular values of the unknown, in the case of an identity, the relationship is true for all values of the unknown. For example, $7x - 5 = 2$ is an equation which is true only when $x = 1$. Submit the value of $x = 1$ into the given equation and check if the left hand side of the equals sign is not equal to the right hand side?

We now turn our turn to linear equations.

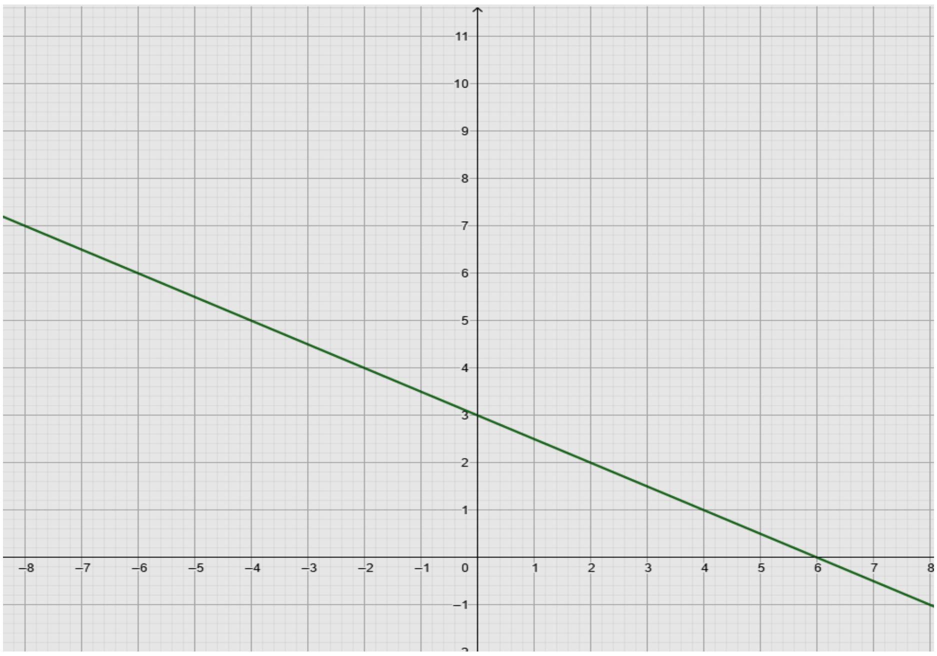
Graphing an Equation in Two Variables by Plotting Points

In order to graph an equation involving two variables, we have to …..

1. Create a three-column table with the first column labelled x, the second column labelled y and the third column being the ordered pairs resulting from the x-y values.
2. Randomly select x-values which are both negative and positive integers.
3. Plot the ordered pairs in the Cartesian plane.
4. Connect the points if they form a line.

E.g. 1. Graph the equation; $y = -x + 3$

x	$y = -x + 3$	(x, y)
-4	$y = -(-4) + 3 = 7$	$(-4, 7)$
-3	$y = -(-3) + 3 = 6$	$(-3, 6)$
-2	$y = -(-2) + 3 = 5$	$(-2, 5)$
-1	$y = -(-1) + 3 = 4$	$(-1, 4)$
0	$y = -(0) + 3 = 3$	$(0, 3)$
1	$y = -(1) + 3 = 2$	$(1, 2)$
2	$y = -(2) + 3 = 1$	$(2, 1)$
3	$y = -(3) + 3 = 0$	$(3, 0)$
4	$y = -(4) + 3 = -1$	$(4, -1)$



From the graph, we notice that the line cuts the y-axis at the point (0, 3) and the x-axis at the point (6,0). The point (0, 3) is referred to as the y-intercept while the point (6,0) known as the x-intercept.

Reflection

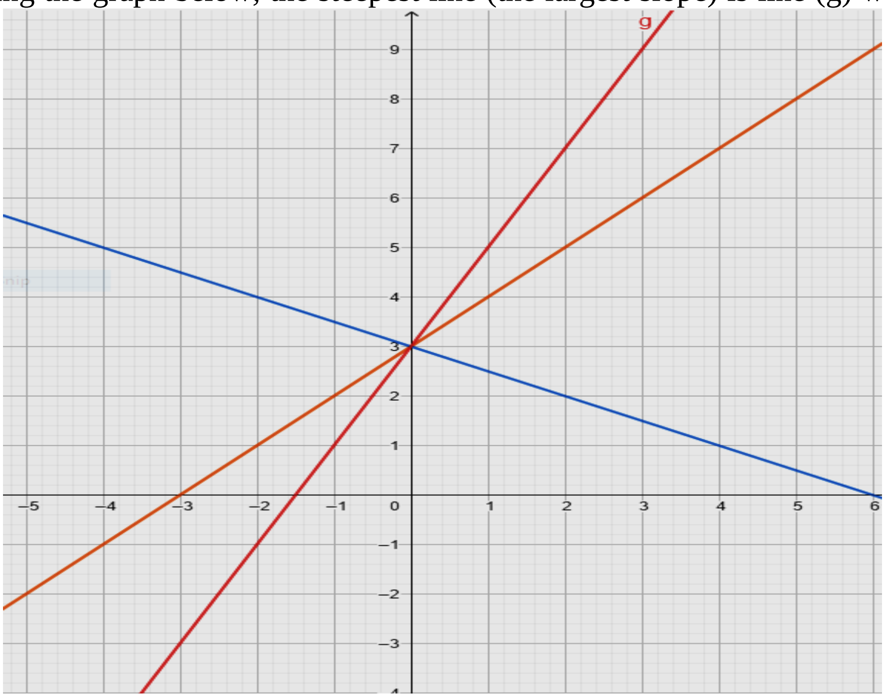
1. Construct a table and graph the equation by plotting points:
   1. $y = \frac{1}{2}x + 2.$
   2. $y = \frac{2}{3}x - \frac{4}{3}.$

In each case, identify both the y-intercepts and the x-intercepts.

2. Slope and Equation of a Line

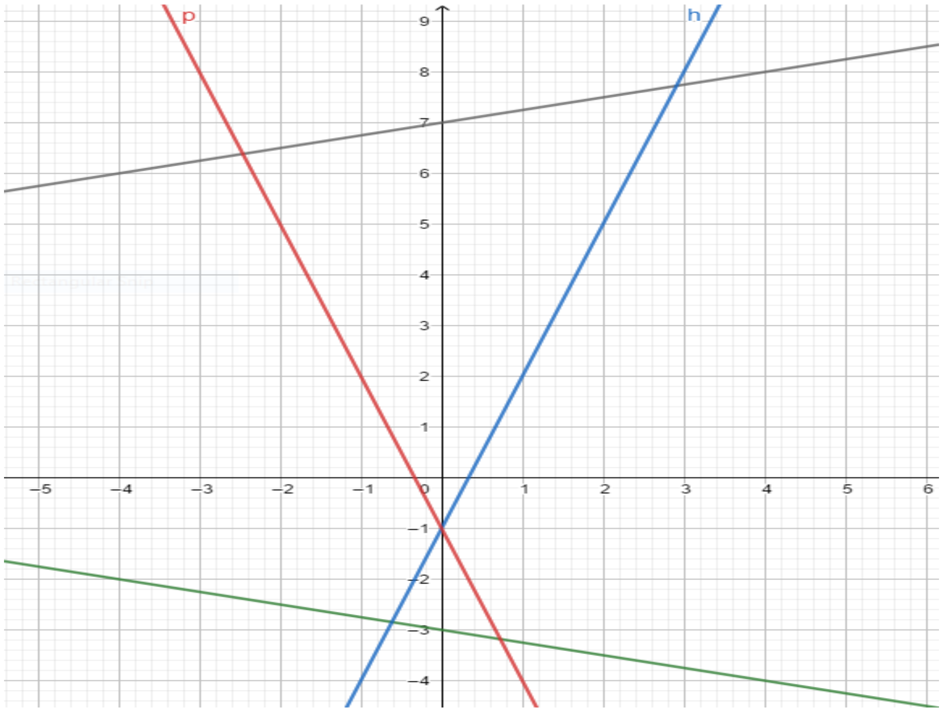
Consider the graph of the following equations: $y = -\frac{1}{2}x + 3$; $y = 2x + 3$ and $y = x + 3$. You'll notice that all the three lines (straight lines or linear equations) have the same y-intercept with coordinates (0, 3); however, they 1) tilt differently and also 2) orient (lie) in different quadrants. Since the only 'thing' that changes in the three equations is the coefficient of x, the slope, it means that the slope of any equation determines two things:

1. **how** the line tilts – the larger the coefficient of x, the steeper the slope
2. **the** sign indicates if the line tilts upwards (e.g. the two lines in quadrants 1 and 3) – in this case, a positive slope- or tilts downwards (e.g. the line in quadrants 2 & 4) – i.e. negative slope. Therefore, using the graph below, the steepest line (the largest slope) is line (g) with equation $y = 2x + 3$.



TRY

Use the following graph to group the lines drawn into those with:



a) positive slopes and;

b) negative slopes.

The slope of a line can be determined through a number of ways such as: reading from the graph, if given the equation of the line, the slope is the coefficient of x, and also by use of a formula. We shall now turn our attention to how to compute the slope of a line using the slope formula.

To determine the slope of a line, we use any two points on the line, e.g. $A(x_1, y_1)$ and $B(x_2, y_2)$. The slope, m $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$. Let's use the equation , $y = -x + 3$ (sub-section 3.2) and the table of values. Choosing the points; $(-4, 7)$ and $(0, 3)$; we substitute these coordinates into the slope formula.

$$m = \frac{7-3}{-4-0} = \frac{4}{-4} = -1$$

$$OR$$

$$m = \frac{3-7}{0-(-4)} = \frac{-4}{4} = -1$$

You will notice that the answer, -1, is the same as the coefficient of x in the given equation. Again, a slope of -1 means that if we increase the x-value by one (1), then we need to decrease the y-value by 1 to get another point on the line. For example, since $(2, 1)$ lies on the line means that $(2+1, 1+(-1)) = (3, 0)$ gives another point that lies on that line.

Reflection

Determine the slope for using the given points: a. $(-3,-2)$ and $(7,-2)$ b. $(6,4)$ and $(6,-1)$ c. $(-4,3)$ and $(2,-8)$

Linear Equations involving One Variable and its applications

Linear equations refer to equations in which the highest degree of the unknown is one. For example, $3x-5=2$ is a linear equation because the unknown (variable) x is of degree one. What does it mean to solve a linear equation?

To solve an equation means to find the value of the unknown (variable) which will make the equation true. In solving for the unknown, any of the basic operations can be used under the condition that 'the equality of the equation is maintained' as we transform the equation into an equivalent forms.

Worked problems involving linear equations in one variable

Example 1. Solve the equation; $3x = 18$	
$\frac{3x}{3} = \frac{18}{3}$	Divide through by 3 so that the coefficient of the unknown, x, is one. By applying the same operation on the left hand side (LHS) and the right hand side (RHS), we obtain an equation which is equivalent to the initial one
$x = 6$	Therefore, the value of the unknown that makes the equation true is 6. Note: You can verify if the value obtained satisfies the equation by substituting it into the original equation and checking if the LHS = RHS. We encourage you to verify and draw your conclusion.
Example 2. Solve the equation; $\frac{-2x}{7} = 6$	
$7\times(\frac{-2x}{7}) = 6\times 7$ $-2x = 42$	Since the LHS is a fraction, we clear the fraction by multiplying both sides of the equation by 7.

$\frac{-2x}{-2} = -\frac{42}{2}$ $x = -21$	To make x the subject, we divide through by -2
Example 3. Solve the equation; $4 - 3p = 2p - 11$	
$4 + 11 = 2p + 3p$	In order to keep the ***p*** term positive, we group like terms keeping all the terms in ***p*** at the RHS and the constant terms to the LHS.
$15 = 5p$	
$\frac{15}{5} = \frac{5p}{5}$ $p = 3$	Divide the LHS and RHS by 5 to form an equivalent equation. Verify if $p = 3$ satisfies the equation.
Example 4. Solve the equation; $\frac{2y}{5} + \frac{3}{4} + 5 = \frac{1}{20} - \frac{3y}{2}$	
$20(\frac{2y}{5}) + 20(\frac{3}{4}) + 5(20) = 20(\frac{1}{20}) - 20(\frac{3y}{2})$	Divide through by the LCM of the denominator which is 20
$4(2y) + 5(3) + 5(20) = 1 - 10(3y)$ $8y + 15 + 100 = 1 - 30y$	
$8y + 30y = 1 - 15 - 100$	Grouping like terms
$38y = -114$	Simplifying
$\frac{38y}{38} = \frac{-114}{38}$ $y = -3$	Further simplification
Example 5. Solve the equation (word problem); A painter is paid GH¢12.00 per hour for working for 36 hours a week, and an overtime paid at one-fourth	We need to form numerical sentence from the given information. That is, Basic rate per hour $=$ GH¢12.00

times this rate. Determine how many hours the painter has to work in a week to earn GH¢513.00	Overtime rate per hour = $= \frac{1}{4} \times \text{GH¢}12.00 = \text{GH¢}3.00$
Let the number of overtime hours worked $= x$ Then, $36 \times 12 + 3 \times x = 513$ $432 + 3x = 513$ $3x = 513 - 432$ $3x = 81$ $x = 27$ Thus, the painter has to work 27 hours overtime in order to earn GH¢513.00. Hence, the total number of hours worked is $36 + 27 = 63$ hours.	

So far we have solved a number of problems involving linear equations in one variable. We hope that you were able to follow the explanations given and can solve the problems on your own?

Reflection

1. Determine the value of x which will make the equation, $\frac{x}{6} + 10 = 15$ true.
2. Twelve workmen employed on a building site earn between them a total of Gh¢2415 per week. Labourers are paid Gh¢175 per week and craftsmen are paid Gh¢220 per week. Determine the number of:
   i. craftsmen and
   i. labourers who were employed.

3. A rectangle has a length of 20 cm and a width **b** cm. When its width is reduced by 4 cm its area becomes 160 cm^2. Find the original width and area of the rectangle.
4. A number exceeds another number by 17 and their sum is 31. Find the two numbers.

So far, we have looked at how to solve linear equations involving one variable algebraically. Let us now turn our attention to how to solve two linear equations still involving one variable. We are still looking for whether the two lines have any point(s) in common. Remember, each equation is a linear function and so a straight line.

Let us consider the following systems of equations:

Worked Example

y=4x-12 --------------------------------------A

y=-3x+2 ---------------------------------------B

We can solve for the value of *x* as follows:

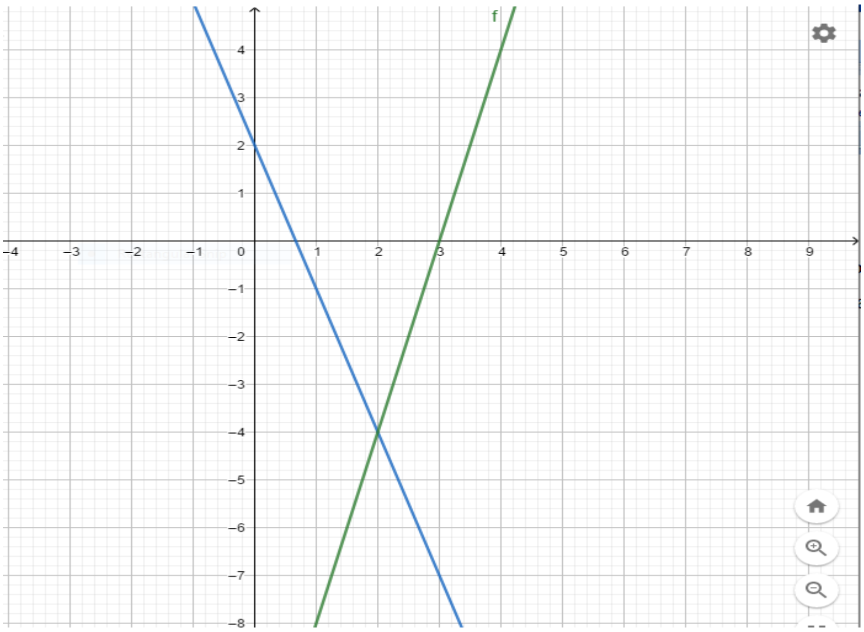
-3x+2=4x-12 -------------------------------------- equating both equations

-3x-4x=-12-2 --------------------------------------grouping like terms

-7x=-14 --------------------------------------simplifying like terms

x=2 ------------------------------------- making *x* the subject

What this means is that if we should substitute x=2 into each of the equations, we shall obtain the same value for *y, i.e. y= -4. The graphs for the two equations are shown below:*



We are sure you have noticed that the two lines intersect only once; when *x* = 2 with their respective *y*-values being -4; the same answer we obtained from the algebraic approach.

Let us take another example.

Worked Example

y=8x-5 --------------------------------------C

y=-4x+7 -------------------------------------D

Since *y* has been made the subject, we can equate both equations and solve for *x* as we did in the first example.

-4x+7=8x-5

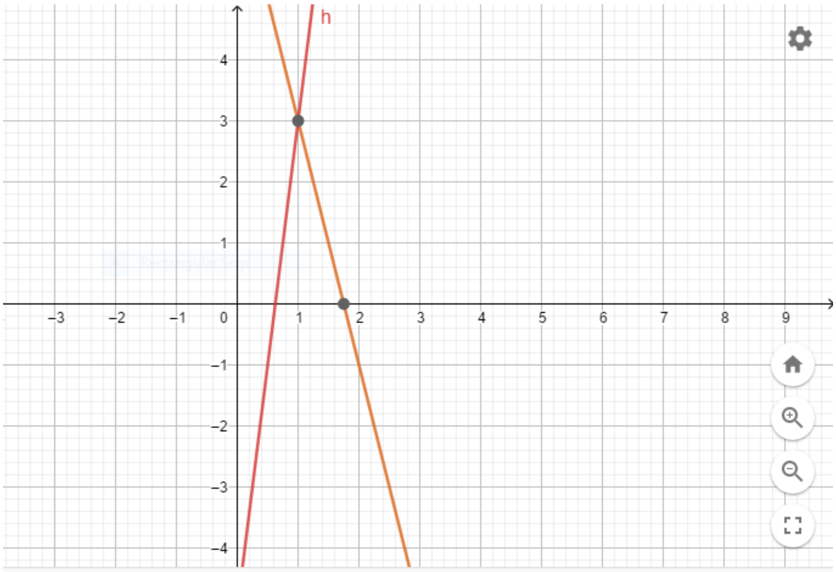
-4x-8x=-5-7

-12x=-12

x=1

From the value for x obtained, this means that when we substitute it into the two equations, we shall obtain the same value for *y*. It must be noted, that this will happen only once, when *x* = 1. From the graphs for the two linear equations, you will notice that the two lines intersect at only once, at *x* =1 and the corresponding *y*-value is 3.

To do: Substitute $x = 1$ into the two equations and verify if you will get $y = 3$ in each case?



So far, we have learned one possibility when solving two linear equations involving one variable. Let us consider another situation using the following example.

Worked Example

$$y=-8x-5$$

$$y=-8x+7$$

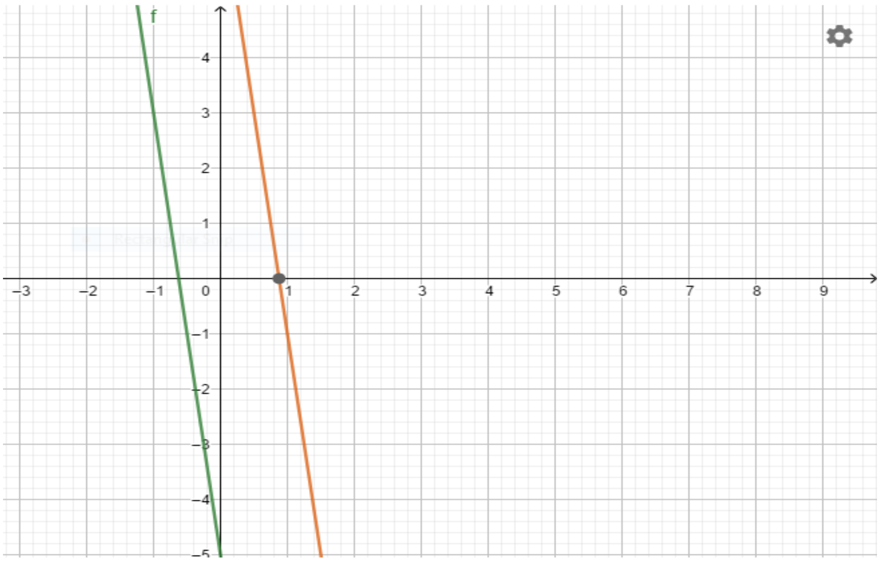
Let us observe what happens when we solve it algebraically?

$$-8x-5=-8x+7$$

$$-8x+8x=7+5$$

$$0=12???$$

You will notice that we obtain a strange answer, 0=12 which we know is not possible. Let us consider the graphs of the two linear equations as shown below.



Did you notice that the two lines do not intersect, in fact, that they are parallel? From the graph, we notice that the two lines are parallel to each other and since we know that parallel lines do not intersect, it means that the given linear equations have no point of intersection hence the strange answer from the algebraic approach. Another thing you would have noticed is from the given equations. You would observe that both linear equations have the same slope, -8. When two or more linear equations have the same slope, then their lines are parallel to each other.

Worked Example

$$y=2x+9$$

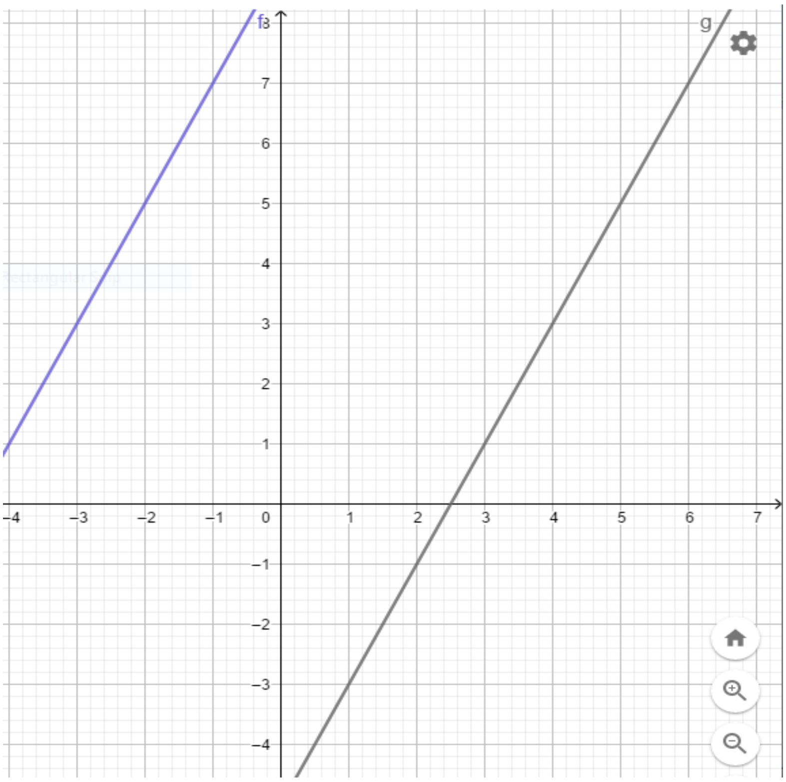
$$y=2x-5$$

$$2x+9=2x-5$$

$$2x-2x=-5-9$$

$$0=??-14$$

In this example too, we observe the same strange answer which is not possible. That should tell us that there is no single value of *x* which will satisfy both equations. Now, let us look at the graphs of the two equations.



From the graphs, we observe that the two lines are parallel; they do not intersect for which reason there is no value of *x* which will satisfy both equations. You would have also noticed that the slopes of the two linear equations which are parallel are the same, however, their y-intercepts are different. Those are the characteristics of parallel lines; same slopes, different y-intercepts.

So far, we have looked at linear equations which have either one solution (the graphs intersect at only one point) or no solution (the graphs do not intersect). Now, let us look at some examples involving cases where there are an infinite number of solutions to the given systems of equations.

Worked Example

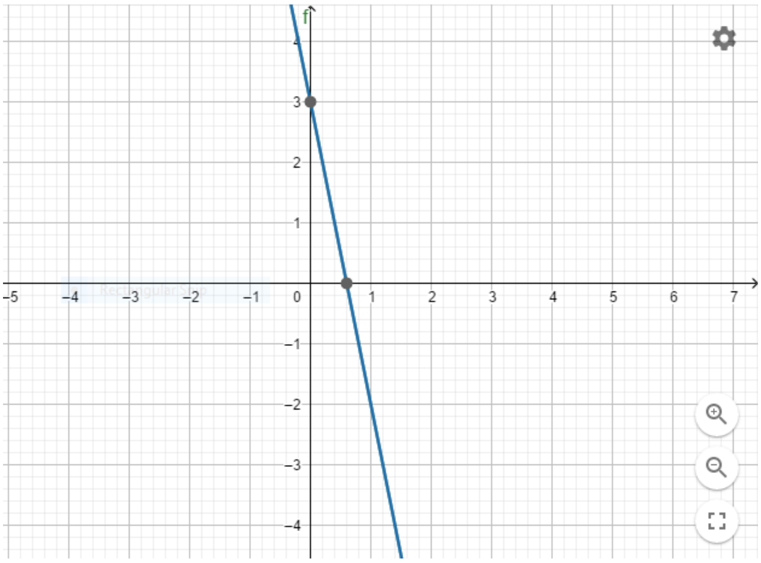
Solve: $4x=3x+x$. In this example, you will notice that for any value of x substituted into the equation, the equation holds true. Try and see for different values of x such as whole numbers, positive and negative integers and fractions.

Worked Example

Solve: $-5x+3=-2x-3x+3$.

Similar to the previous example, try to substitute different values for *x* and verify if the equation holds true. Hopefully, you realised that irrespective of the value of x, the equation is true. As such, the solution set consists of all values that make the equation true giving as an infinite solution.

In such cases, the graphs of the two equations are the same; one graph lying on top of the other as shown below for the equations:$y=-5x+3$ and $y=-2x-3x+3$:



In summary, we can classify linear equations into three possibilities:

a. a **conditional** equation – which has only one solution point (their graphs intersect at only one point),

b. an **inconsistent** equation – which results in a false statement (no solution – graphs produce parallel lines) ,

c. an **identity** equation – which is true for all values of the variable (infinitely many solutions – graphs produce a single line).

Reflection

Solve the following equations:	
a.	$2x+7=19$
b.	$6[4-27y-1]=8(13-8y)$
c.	$12[1-54z-1]=3(24+11z)$
d.	$4x-3+12=15-5(x+6)$
e.	$72x-53x=223$
f.	$2x-32=72x$
g.	$0.36100n+5]=0.6(30n+15)$

SESSION 3: LINEAR EQUATIONS INVOLVING TWO VARIABLES AND ITS APPLICATIONS

In the previous session, we learned how to formulate and solve linear equations involving one variable. However, in real life there are instances where you may require two or more variables to accurately describe the situation. In this session, we shall consider systems of equations in two variables and how to solve them.

Learning outcomes

By the end of the session, the participant will be able to:

1. find the solution for a system of linear equations and,
2. formulate a system of linear equations that models a given context.

A system of equations refers to two or more linear equations made up of two or more variables. In such cases, solving the system of equations require that all the equations are solved simultaneously. We shall look at the situation where the systems of linear equations have two different variables; that is, two equations with two variables.

Similar to the examples we considered in Session 2, such systems of equations may or may not have a solution. When we talk about a **solution,** we are referring to any ordered pair (x, y) that satisfies each equation independently. For example, given:

$$x+y=4$$

$$3x-y=0$$

Solving simultaneously, we have: $4x=4$

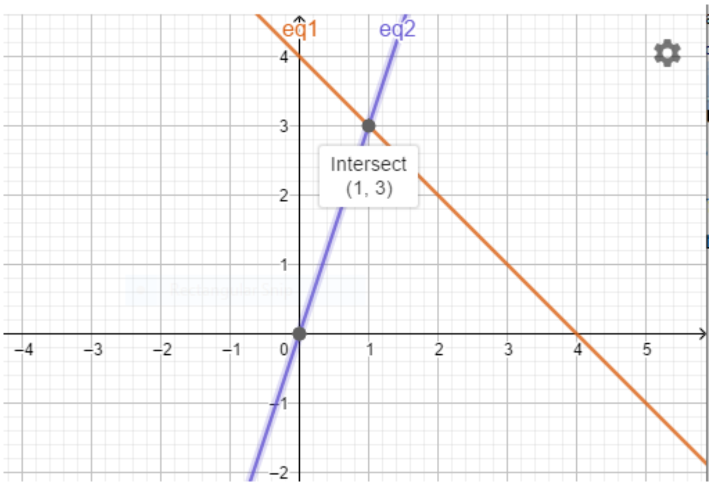
$$x=1$$

Substitute the value of $x=1$ into any of the two equations and solving for y, we have:

$$1+y=4$$

$$y=4-1=3$$

Therefore, the solution for the given systems of equation is (1, 3). We can confirm this solution from the graphs for the two linear equations below:



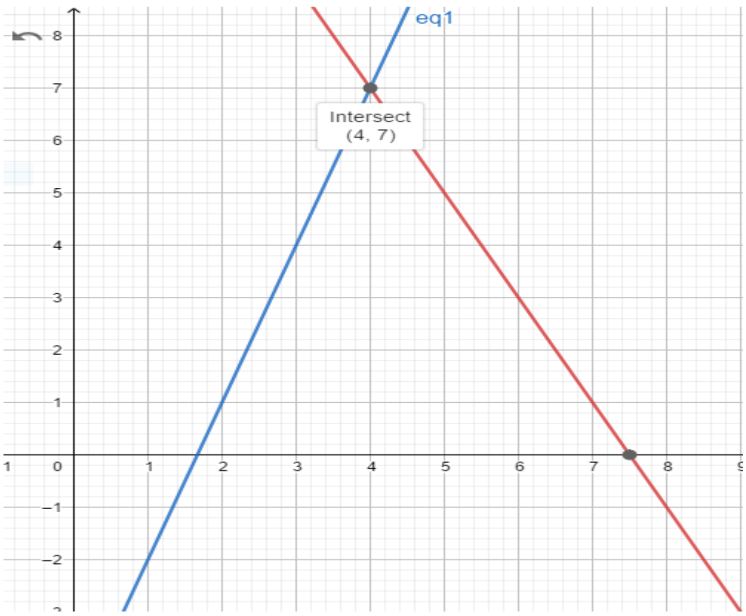
Verify that the point (1, 3) satisfies the two equations independently.

Worked Example

Solve: $2x+y=15$

$3x-y=5$

If you obtained the solution (4, 7), that is correct! Well done! This solution is confirmed from the graphs of the two equations below.



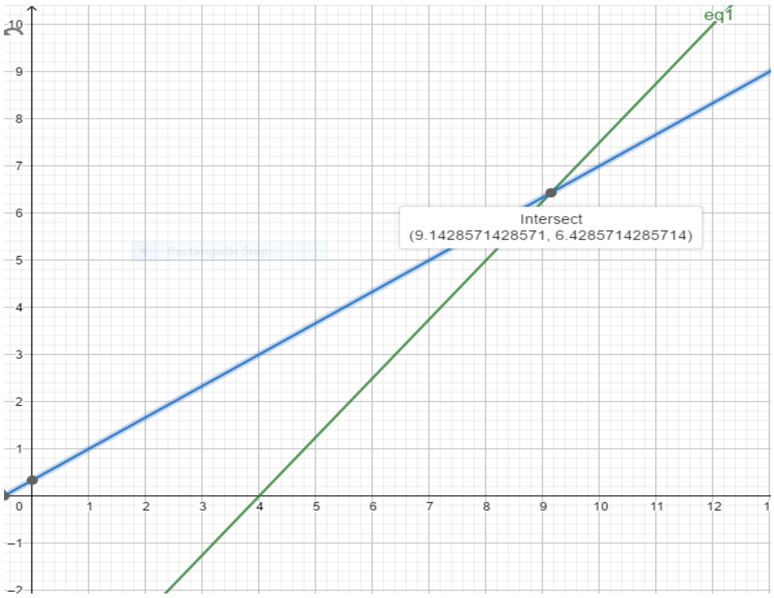
Worked Example

Determine whether the ordered pair (8, 5) is a solution to the following systems of equation:

$5x-4y=20$

$2x+1=3y$

There are a number of ways of determining whether the given point is a solution such as use of the graphical approach. From the graph below, you would notice that the two graphs intersect at only point hence there is a solution. However, the solution is not the given point (8, 5) rather, (9.1 , 6.4). Therefore, the point (8, 5) is **not** a solution to the given systems of linear equations.



Another method is to substitute the given point into the two equations to verify if the equation is true in each case.

From $5x-4y=20$ where $x=8$ and $y=5$; we have :

$$5(8)-4(5)=20$$

$$40-20=20 \text{ as required.}$$

Now, let's check for the second equation: $2x+1=3y$

$$2(8)+1=3(5)$$

$$17\neq 15$$

We notice that while the point (8, 5) satisfies the first equation, it does not hold true for the second equation. Therefore, the point (8, 5) is **not** a solution for the systems of equation which is the same conclusion we came to from observing the graphs.

Similar to how we were able to classify the linear equations in one variable depending on the nature of solution(s), we can do same for systems of linear equations in two variables. Basically, there are three types of systems of linear equations in two variables and three types of solutions.

1. An **independent** system: This type of solution has exactly one solution which is an ordered pair (x, y). From its graphs, the two lines intersect only once which is the only solution for the two linear equations. This is the case of our first solved example in this session.
2. An **inconsistent** system: This type of systems of linear equations in two variables has **no** solution. If you are to graph the two equations, you will have parallel lines which never intersect. For example:

   Solve: $2x+4y=26$

   $x+2y=10$

We can rewrite these two equations in the slope- intercept form as follows:

$$y= 12x+264$$

$$y= 12x+102$$

By rewriting the equations in this form, we observe that the two equations have the same slope but different y-intercepts. Now let us solve it algebraically.

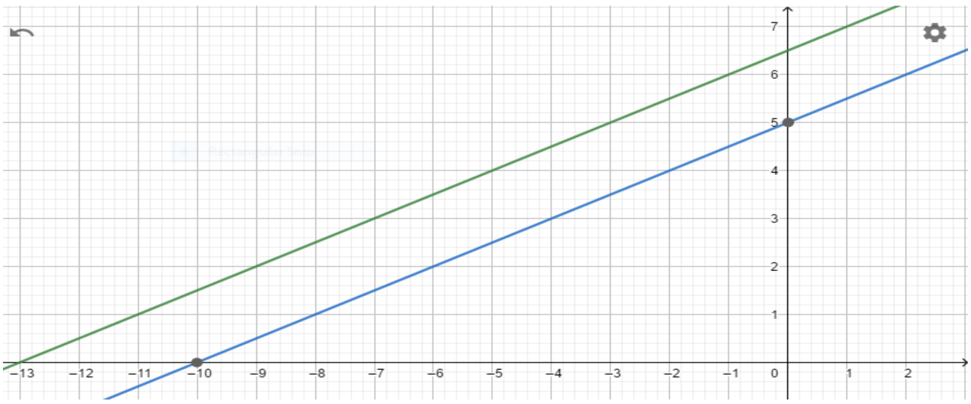
$$12x+102= 12x+264$$

$$12x-12x=264-102$$

$$0=32$$

When we solve the two linear equations we obtain a false statement (contradiction) because $0\neq32$. Therefore, the system has **no** solution

So from the algebra, we know that the system has no solution, an inconsistent system of equations. Let us now look at their graphs as shown in the diagram below.



We observe that the lines are parallel to each other which also means that there is no solution as expected.

3. A **dependent** system: This type of systems of linear equations in two variables has **infinitely many** solutions. If you are to graph the two linear equations in two variables, you shall have the same line (the lines are coincident) which means that every ordered pair (x, y) on the line is a solution to both equations. For example,

$x+2y= 2$ ______________________(1)

$2x+4y= 4$ ______________________(2)

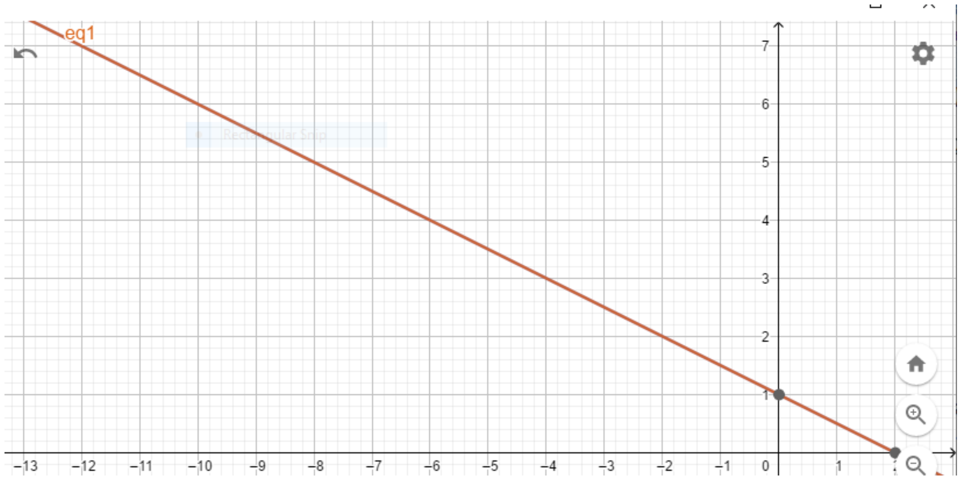
Let us first of all solve the system algebraically.

Equation (1) x 2: $2x+4y= 4$

$2x+4y= 4$

Subtracting: $0= 0$

From this algebraic answer, we can see that there will be an infinite number of solutions that satisfy both equations. Try using different numbers in the Real Number system and see. Also, try and graph the two equations on the same graph. Did you get the same line? Then you are on track. That is what is expected since the two equations are the same hence produce the same line (or one line on top of the other) as shown in the diagram below.



A careful observation of the two equations will show that one equation is a multiply of the other. This is characteristic of consistent systems

$x+2y=2$ ________________________(1)

$2x+4y=4$ _________________________(2)

Worked Example:

A movie theatre charges GHȻ 4.00 for each child's ticket and GHȻ 6.50 for each adult's ticket. One night 200 tickets were sold resulting in an amount of GHȻ1,100.00 in ticket sales. How many :

a. children's tickets were sold?
b. adults' tickets were sold?

Let x represent the number of children's tickets sold and y, the number of adult tickets sold.

$x + y = 200$ (equation for total number of tickets sold) _______________________1

$4x + 6.50y = 1100$ (equation for amount realised) _______________________2

Making x the subject in equation (1) : $x = 200 - y$ _________________________3

$4(200 - y) + 6.50y = 1100$ (Substitute $x = 200 - y$ into equation 2)

$800 - 4y + 6.50y = 1100$

$y = 120$

$x = 200 - y = 200 - 120 = 80$

Therefore, 80 children's tickets were sold, and 120 adult tickets were sold.

Your turn!

The cost of a ticket to watch a play at a theatre is GH₵25.00 for children and GH₵50.00 for adults. On a certain day, attendance at the theatre is 2,000 and the total gate revenue is GH₵70,000. How many children and how many adults bought tickets?

Hopefully, you got the following answers: number of children who bought the tickets is 1,200 and number of adults who bought the tickets is 800?

Worked Example

Store clerk sold 60 pairs of sneakers. The high-tops sold for GH₵98.99 and the low-tops sold for GH₵129.99. If the receipts for the two types of sales totalled GH₵6,404.40, how many of each type of sneaker were sold?

Let *h* represent high-tops and *l* represent low-tops

$h+l=60$ ___________(1)

$98.99h+129.99l=6404.40$ _____(2)

Making *h* the subject in equation (1) : $h=60-l$ _______________(3)

Substitute equation (3) into equation (2).

$$98.99(60-l)+129.99l=6404.40$$

$$5939.4-98.99l+129.99l=6404.40$$

$$31l=465$$

$$l=15$$

Substitute $l=15$ into equation (3): $h=60-15=45$

Therefore, number of High-tops is 45 and number of Low-tops is 15.

Reflection

Solve the following equations:

Solve the following systems of equations:

1) a) $2x + 3y = 1$
   $x - 2y = -3$

   b) $-3x + 2y = 12$
   $4x + 2y = -2$

2) Determine whether the given ordered pair is a solution to the system of equations.

   a) $5x - y = 4$
   $x + 6y = 2$ and $(4, 0)$

   b) $-3x - 5y = 13$
   $-x + 4y = 10$ and $(-6, 1)$

   c) $3x + 7y = 1$
   $2x + 4y = 0$ and $(2, 3)$

d) $-2x + 5y = 7$
$2x + 9y = 7$ and $(-1, 1)$

3) Solve each system by substitution.

a) $x + 3y = 5$
$2x + 3y = 4$

b) $3x - 2y = 18$
$5x + 10y = -10$

c) $4x + 2y = -10$
$3x + 9y = 0$

d) $2x + 4y = -3.8$
$9x - 5y = 1.3$

4) A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x$. Find the break-even point.

5) A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was GH₵12.50, and the price for an adult ticket was GH₵16.00. The register confirms that GH₵5,075.00 was taken in. How many student tickets and adult tickets were sold?

6) Admission into an amusement park for 4 children and 2 adults is GH₵116.90. For 6 children and 3 adults, the admission is GH₵175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

SESSION 4: QUADRATIC EQUATIONS INVOLVING ONE VARIABLE (BY GRAPHING, FACTORISATION AND FORMULA)

The previous three sessions focused on linear functions and how to solve linear equations. In this session we shall look at Quadratic Functions and Equations involving one variable. Specifically, we shall consider the characteristics of quadratic functions and how to solve quadratic equations using three different methods namely: factorisation, formula and graphing.

Learning outcomes

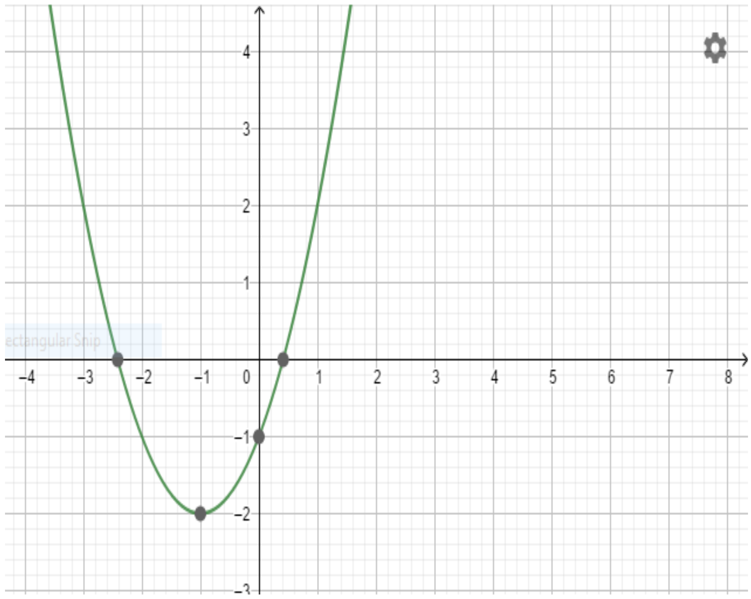
By the end of the session, the participant will be able to:
1. identify the characteristics of the graphs of quadratic functions,
2. solve given quadratic equations using factorisation,
3. solve given quadratic equations using the formula,
4. solve given quadratic equations using graphical approach.

Quadratic functions and their equations are very useful in solving everyday problems involving area, projectile motion (describing the trajectory of a ball, determining the height of a ball thrown upwards) and in optimizing profit for businesses.

Characteristics of Quadratic Graphs

A quadratic equation in x is an equation of the form ax2+bx+c=0, where a, b, and c are constants. Quadratic equations are polynomials of degree two such as 2x2-7x+4=0 and x2-9=0.

The graph of a quadratic function is called a parabola. This graph is a **U-shaped curve if it opens upwards** and ∩**-shaped if it opens downwards**. Parabolas have the following features: a) vertex – which represents the extreme point on the curve. If you have a U-shaped curve, the vertex is the lowest point on the graph or the minimum value of the quadratic function. If the graph is ∩-shaped the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry.**

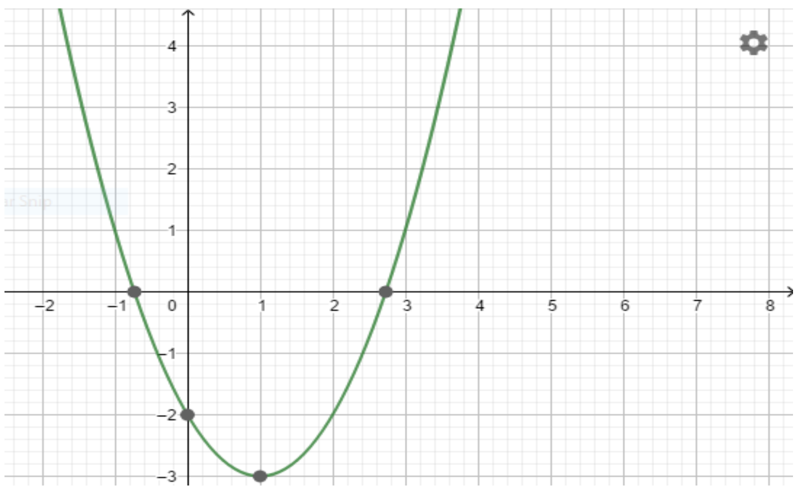


From the graph above, the features described can be identified: a) vertex is at negative 2 (i.e. -2);

b) *x*-intercepts are -2.5 and 0.5; c) *y*-intercept is negative 1 (i.e. -1); d) the axis of symmetry passes through the vertex at negative 2.

The y-intercept is the point at which the parabola intercepts the *y*-axis. The *x*-intercepts are the points at which the parabola intercepts the *x*-axis. If the *x*-intercepts exist, they represent the **zeros** or **roots** of the quadratic function. The zeros of a quadratic function give the values of *x* at which $y = 0$.

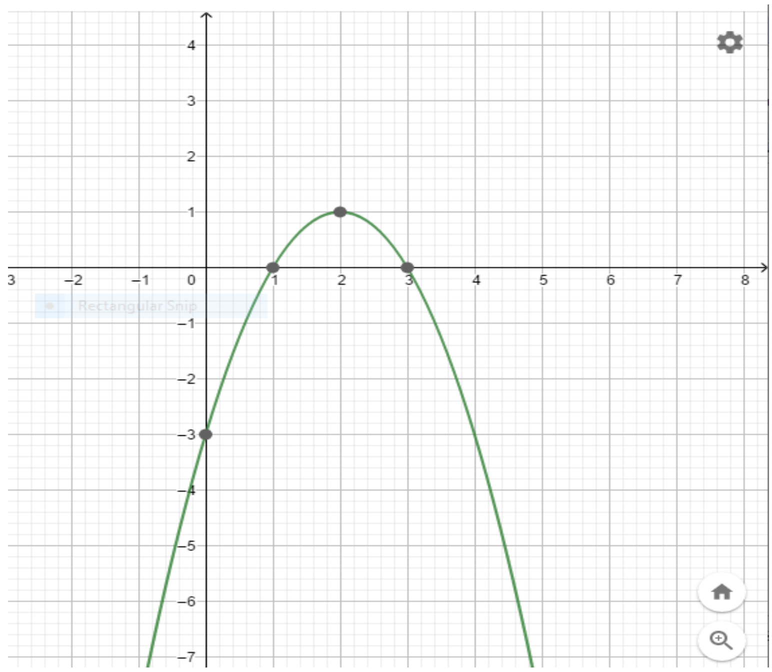
Worked Example



The vertex is the turning point of the graph. We can see that the vertex is at (1, 3). Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 1$. This parabola intersects the x-axis at $x = -0.8$ and $x = 2.8$ and they are the zeros of the quadratic function. It crosses the y-axis at (0, -2) so this is the y-intercept.

Worked Example

Identify the features of the graph shown below.



The vertex is the turning point of the graph. We can see that the vertex is at (2, 1). The axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 2$. Note that since the parabola opens downwards, the vertex is the maximum point on the curve. This parabola intersects the x-axis at $x = 1$ and $x = 3$ and they are also the zeros of the quadratic function. It crosses the y-axis at (0, -3) so this is the y-intercept.

Solving Quadratic Equations by Factoring

The expression x-3(x+3) can be multiplied out to give x2-9=0 and this reverse process is called **factorisation**. The process of factorisation means finding expressions that can be multiplied together to give the expression on one side of the equation. However, it is not all expressions that can be factorised. If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a \cdot b = 0$, then $a = 0$ or $b = 0$, where a and b are real numbers or algebraic expressions. What this means is that, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

When you multiply the factors, you expand the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication reverses the operation of factoring thereby giving you the initial (quadratic) equation. For example, we can expand x-3(x+3) by multiplying the two factors as follows:

x-3x+3=xx+3-3x+3

$= x^2+3x-3x-9$

$= x^2-9$

By expanding the two factors, we end up with a product, which is a quadratic expression. If we should set this expression to zero, $x^2-9=0$ we obtain a quadratic equation and if we were to factor the quadratic equation, we would get back the factors we multiplied.

Remember that the general form of a quadratic equation is $ax^2+bx+c=0$, where *a*, *b*, and *c* are constants. The processes for factorising a quadratic equation are as follows:

1. Find two numbers which multiply to give *c* and whose sum equals *b*.

2. Use those numbers to write two factors of the form $(x + k)$ or $(x - k)$, where *k* is one of the numbers found in step 1.

Use the numbers exactly as they are. In other words, if the two numbers are 5 and −3, the factors are

$(x + 5)(x - 3)$.

3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

Worked Example

Factorise and solve the equation: x^2+x-6

To factor x^2+x-6

Step 1: we look for two numbers whose product equals −6 and whose sum equals 1. The possible factors of −6 are: 1 x (−6); (−6)x 1; 2 x (−3); 3 x (−2)

Step 2: The pair, 3 x (−2) sums to 1 which is *a (the coefficient of* x^2) so these are the numbers of interest. Note that only one pair of numbers will work.

Step 3: We write the factors as follows: $(x - 2)(x + 3) = 0$

To solve this equation, we use the zero-product property by setting each factor equal to zero.

$$(x - 2)(x + 3) = 0$$

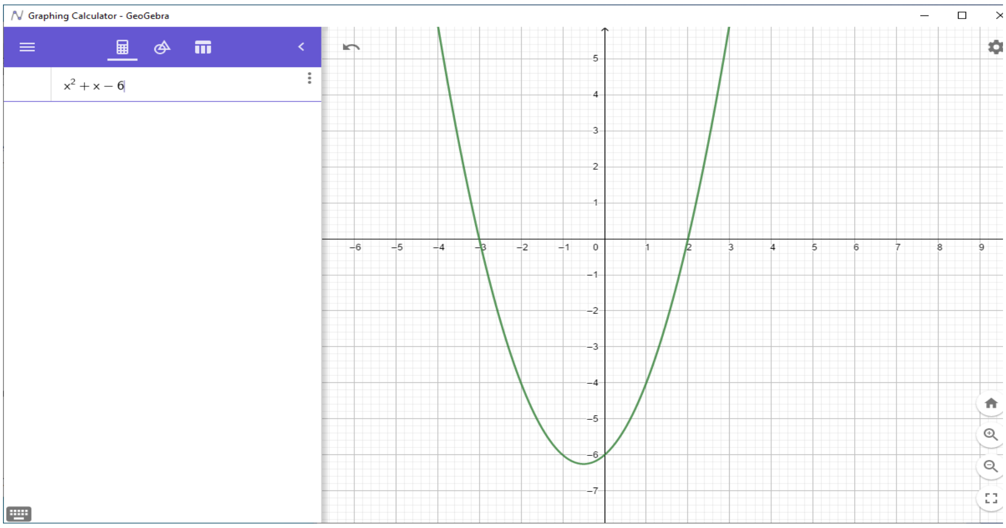
$$(x - 2) = 0$$

$$x = 2$$

$$(x + 3) = 0$$

$$x = -3$$

If we are to graph the given quadratic equation, we would notice that the solutions to the equation are the x-intercepts (roots or zeros) as shown below.



From the graph, we observe that the zeros are x= -3 and x = 2 which are solutions to the quadratic equation we obtained using the method of factorization.

Worked Example

Factor and solve the quadratic equation:

$$x2\text{-}5x\text{-}6=0$$

We have $x+1(x\text{-}6)=0$ because 1x (-6) = -6 and -6x + x = -5x.

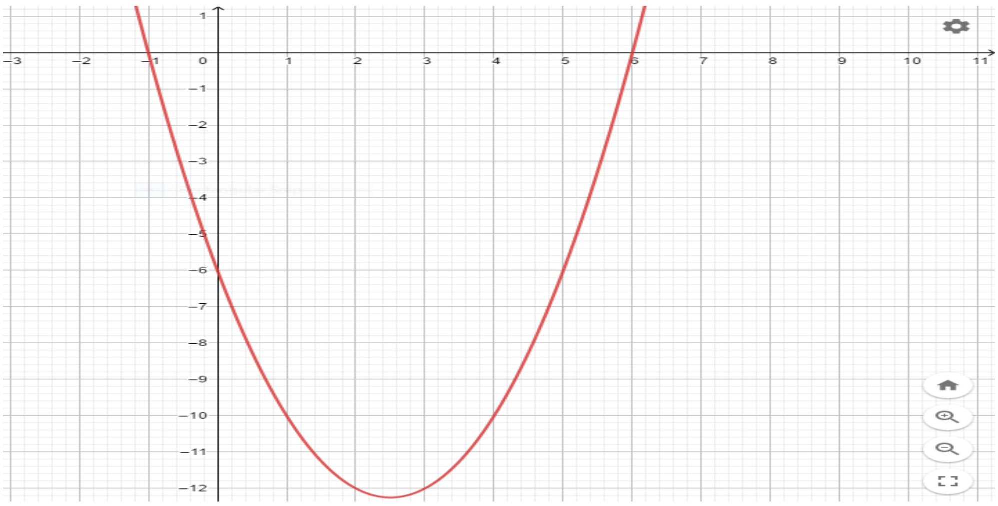
Solving for x: $x+1=0$

$$x=-1$$

$$(x-6)=0$$

$$x=6$$

The solutions to the equation is also confirmed from the graph of the equation shown below.



So far, we have factorised quadratic equations where the coefficient of x^2 is 1. We shall look at how to factorise quadratic equations where the coefficient of x^2 (leading coefficient) is greater than one. *The processes are as follows:*

When the leading coefficient is not 1, we factor a quadratic equation using the method called grouping, which requires four terms.

1. With the quadratic equation in standard form, ax2+bx+c=0,, multiply *a* x *c*.
2. Find two numbers whose product equals *a x c* and whose sum equals *b (coefficient of the term in x)*.
3. Rewrite the equation replacing the *bx* term with two terms using the numbers found in step 1 as coefficients of *x*.
4. Factor the first two terms and then factor the last two terms. The expressions in brackets or parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

Worked Example

Factorise and solve: 4x2+15x+9=0.

Step 1: Multiply *a* x *c: 4 x 9 = 36.* Then find the factors of 36 which are : 1x 36; 2 x 18; 3 x12; 4 x 9 ; 6 x 6; -1 x -36; -2 x -18; -3 x -12; - 4 x - 9 ; -6 x -6.

Step 2: Look for the pair of factors whose sum gives 15, that is, the coefficient of *x* which is 3 +12.

Step 3: Rewrite the equation replacing the *bx* term, 15*x*, with two terms using 3 and 12 as coefficients of *x*.

Step 4: Factor the first two terms, and then factor the last two terms.

4x2+12x+3x+9=0.

4x(x+3)+3(x+3)=0.

4x+3(x+3)=0.

Step 5: Solve using the zero-product property.

4x+3(x+3)=0.

4x+3=0.

4x=-3.

x= -34

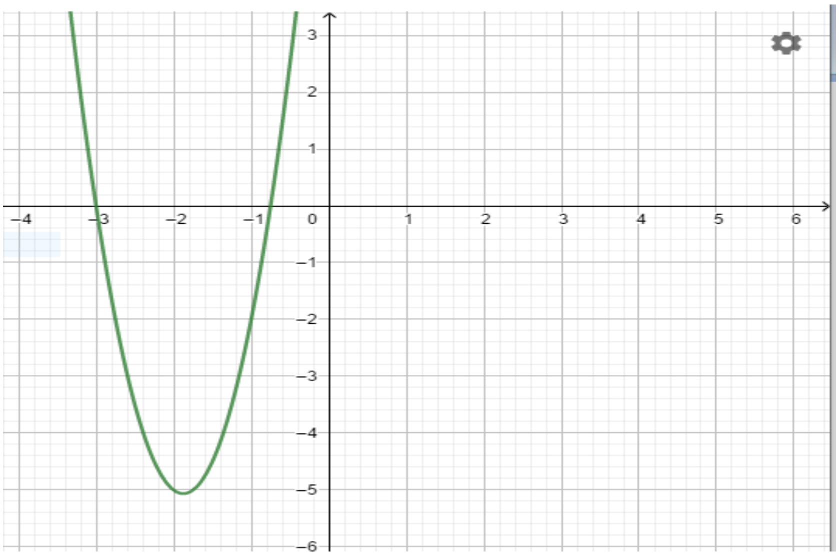
OR

(x+3)=0.

x=-3.

The solutions are -34 and -3.

We can confirm these solutions from the graph below.



Try:

Factorise and solve: -3x2-13x+10=0
Step 1: Did you get the two numbers to be 5 and -2 (for the constant, *c*) and *x* and -3*x* (for the coefficient of x^2).
Step 2: Form the two brackets which contain the factors for the given equation.

-3x2-13x+10=5+x(2-3x)

Hopefully, you got the solutions to be -5 and 23 ?

Reflection

	1. Factorise and solve each of the following equations:
a.	x2-3x+2=0.
b.	x2+7x+12=0.
c.	3x2-7x+2=0.
d.	-25x2-15x+54=0.
e.	4x2+3x.
f.	12x2+11x+2=0.
g.	2x2-x-3=0.

Solving Quadratic Equations using the Formula

So far, we have looked at how to solve quadratic equations using the method of factorisation and graphing. However, the factoring approach only works when the given equation is factorable. The formula to be used is capable of being used to determine the solutions to any quadratic equation.
The formula, =-b±b2-4ac2a, a≠0, often called quadratic formula is used to solve quadratic equations. Here the steps to follow when you are using the quadratic formula to solve quadratic equation:

1. Ensure that the quadratic equation is written in standard form: ax2+bx+c=0.
2. Identify the values of the coefficients and constant term, a, b, and c.
3. Substitute the values noted in step 2 into the equation. Use brackets around each number to be substituted into the formula to avoid unnecessary errors.
4. Perform the necessary manipulations.

Worked Example

Solve the quadratic equation: x2-6x+9=0 where *a* =1, *b* = -6, *c* = 9

From x=-b±b2-4ac2a, a≠0

x=-(-6)±(-6)2-41(9)2(1)

x=6±36-362

x=6±02

x=62=3

Worked Example

Solve the quadratic equation: 4x2+15x+9=0

4x2+15x+9=0 where *a* =4, *b* = 15, *c* = 9

x=-(15)±(15)2-44(9)2(4)

x=-15±225-1448

x=-15±818

x=-15±98

x=-15+98= -68= -34

x=-15-98= -248= -3

Therefore, x= -34 and -3.

Worked Example

Solve the equation: -3x2-13x+10=0 , where *a*= -3, *b* = -13, *c* = 10

x=-(-13)±(-13)2-4-3(10)2(-3)

x=13±169+120-6

x=13±289-6

x=13±17-6

x=13+17-6= 30-6= -5

x=13-17-6= -4-6= 23

Therefore, x= -34 and -3.

Reflection

Solve the following quadratic equations using the quadratic formula.

a. x2+4x+4=0
b. 8x2+14x+3=0
c. 3x2-5x-2=0
d. 3x2-10x+15=0
e. 9x2-46x+25=0
f. 2x2-7x+4=0

UNIT 8: LINEAR INEQUALITIES

In this unit, we introduce participants to linear inequations. We will create contexts for linear inequalities and guide participants to solve simple real-life problems on linear inequalities.

Learning outcome(s)
By the end of the unit, the participant will be able to:
1. solve simple linear inequalities
2. graph linear inequalities
3. solve word problems involving linear inequalities

SESSION 1: LINEAR INEQUALITIES INVOLVING ONE VARIABLE

So far, we have looked at both linear and quadratic equations. In this session, we shall learn about linear inequalities in one including how to represent the solutions to an inequality.

Learning outcomes

By the end of the session, the participant will be able to:

1. use interval notations to represent solutions to inequalities.
2. solve inequalities in one variable algebraically

Use interval notations to represent solutions to inequalities

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

Firstly, we can use a number line as shown below. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

Secondly, we can use the set-builder notation: $x{:}x \geq 4$ which reads: "all real numbers *x* such that *x* is greater than or equal to 4." Notice that braces are used to indicate a set.

Thirdly, we can use the **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. Note that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equalled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-3, 8)$, or all numbers between -3 and 8, including -3, but **not** including 8; $(-7, 0)$, all real numbers between, but **not** including -7 and 0; and $[7, \infty)$, all real numbers greater than and including 7.

Solve inequalities in one variable algebraically
Let us take some examples to illustrate how to solve inequalities algebraically.

Worked Example

Solve the inequality: $x-4>2$

$x-4>2$

$x-4+4>2+4$ (adding 4 to both sides)

$x>6$

Worked Example

Solve the inequality: 13b+2<1

13b+2<1

31 3b+2<31 multiply both sides by 3

b+6-6<3-6 subtract 6 from both sides

b<3-6

b<-3

In notation form, we have: $(-\infty, -3)$

Worked Example

Solve the inequality: $13-7a\geq 10a-4$

$13-7a\geq 10a-4$

$13+4\geq 10a+7a$ (group like terms)

$17\geq 17a$

1717≥ 17a17 (divide through by 17 to make a the subject)

$1\geq a$ or $a\leq 1$

In notation form we write: $(-\infty, 1]$

Worked Example

Solve the following inequality and write the answer in interval notation: -34x ≥ -58+ 23x.

-34x-23x≥ -58+ 23x-23x (group like terms)

-34x-23x≥ -58

-912x-812x≥ -58 (LCM for the right side of the inequality)

-1712x≥ -58 (simplifying the R.H.S fraction)

x≤ 1534.

In notation form, we have: (-∞, 1534].

Reflection

1. Solve the following inequality and illustrate its solution on a number line: $34x+2<x+1$.
2. Solve the following inequality and write its solution in interval notation:
   $-56 \leq 34+ 83x$.
3. Solve the following inequalities:
   a. $3\leq 2x+2<6$
   b. $3+x>7x-2>5x-10$.

SESSION 2: GRAPHING OF LINEAR INEQUALITIES

In this session, we shall learn about how to graph solve linear inequalities and use this approach to solve linear inequalities involving two variables. We shall begin with a review of the Cartesian Plane.

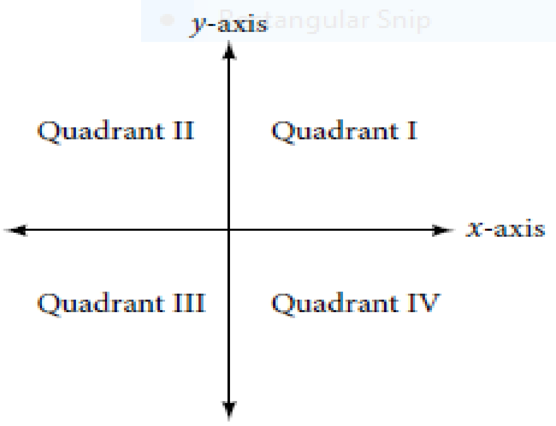
Learning outcomes

By the end of the session, the participant will be able to:

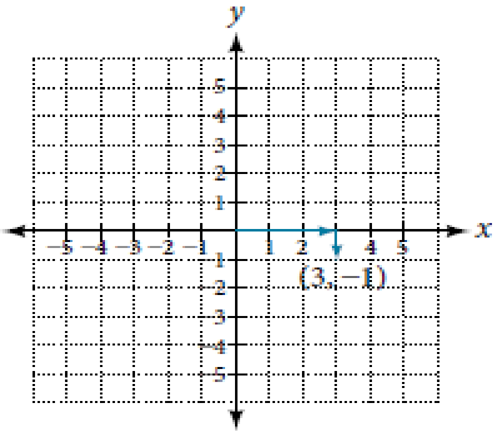
1. Use interval notations to represent solutions to inequalities.
2. Solve inequalities in two variables graphically.

Points on the Cartesian Plane

The points that we represent on the number line can also be represented on a plane. On the plane, each point is the graph of an **ordered pair** (x, y).
A **Cartesian plane** or **Coordinate plane** consists of two perpendicular number lines called axes. The vertical axis is called the **y-axis and** the horizontal axis is called the **x-axis**. Both the x-axis and y-axis intersect at a point called the origin, O. From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the *x*-axis and up the *y*-axis; decreasing, negative numbers to the left on the *x*-axis and down the *y*-axis. That is, the origin divides the plane into four regions or quadrants as shown below.

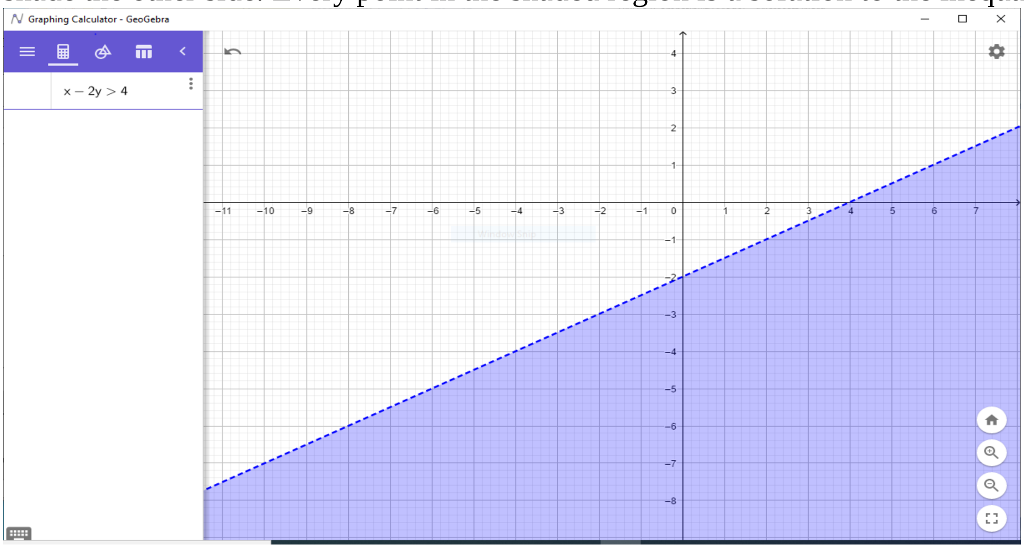


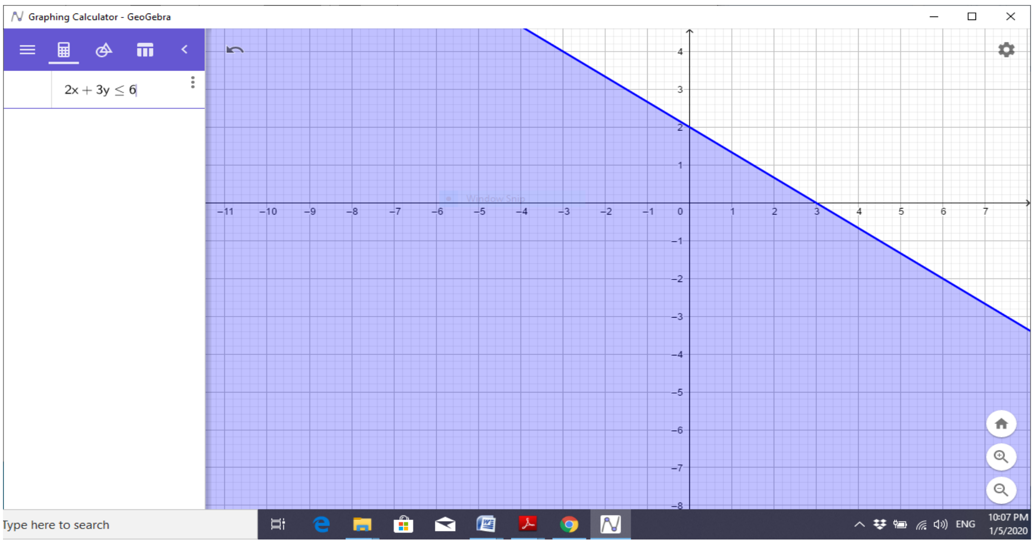
The axes extend to positive and negative infinity as shown by the arrowheads. Each point in the plane is identified by its ***x*-coordinate**, or horizontal displacement from the origin, and its ***y*-coordinate**, or vertical displacement from the origin. Together, we write them as an **ordered pair** indicating the combined distance from the origin in the form (x, y). An ordered pair is also known as a coordinate pair because it consists of x - and y -coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction.

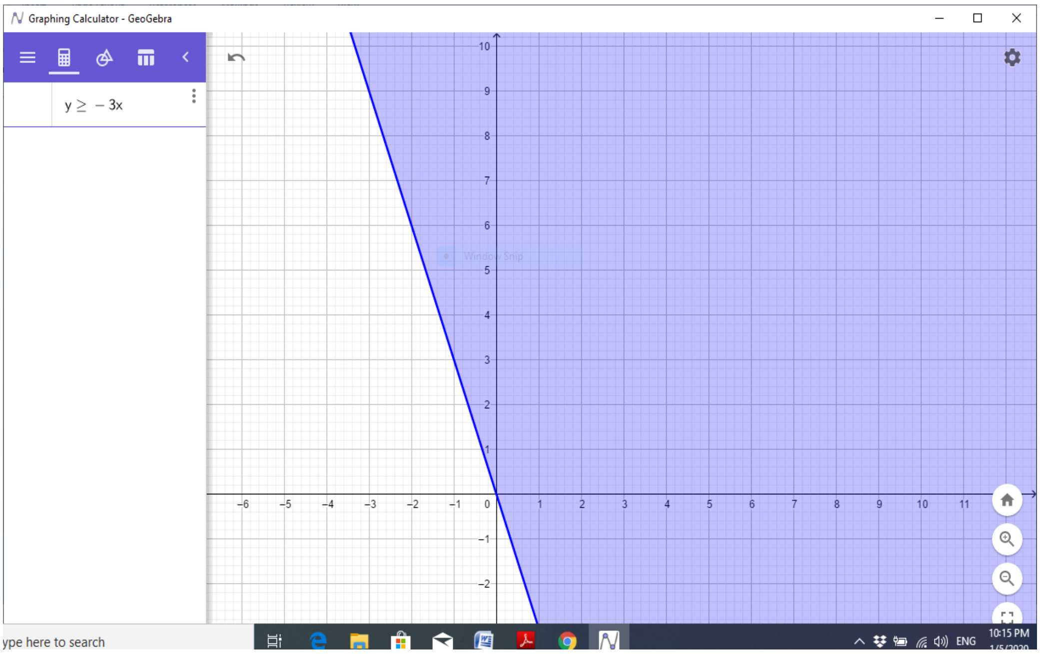


Graphing of Linear Inequalities

When sketching the graph for an inequality, we will use a solid graph for "≤" and "≥" inequalities, and a dashed graph for "<" and ">" inequalities. We can decide which side of the graph to shade by choosing any point not on the graph itself. We will put this point into the inequality. If it makes the inequality true, we will shade the side that has that point. If it makes the inequality false, we will shade the other side. Every point in the shaded region is a solution to the inequality.







Graphical Solution of Systems of Linear Inequalities

If the solution for a given system of inequalities exists, it is usually a region in the plane. The solution to a polynomial inequality is the region above or below the curve.
Here the steps to follow when graphing inequalities;

a. Draw the first inequality. The line for this inequality divides the coordinate plane into two regions. One part of the region satisfies the inequality.

b. Using the coordinates of the point on one side of the line, determine the region satisfied by the inequality and shade it.

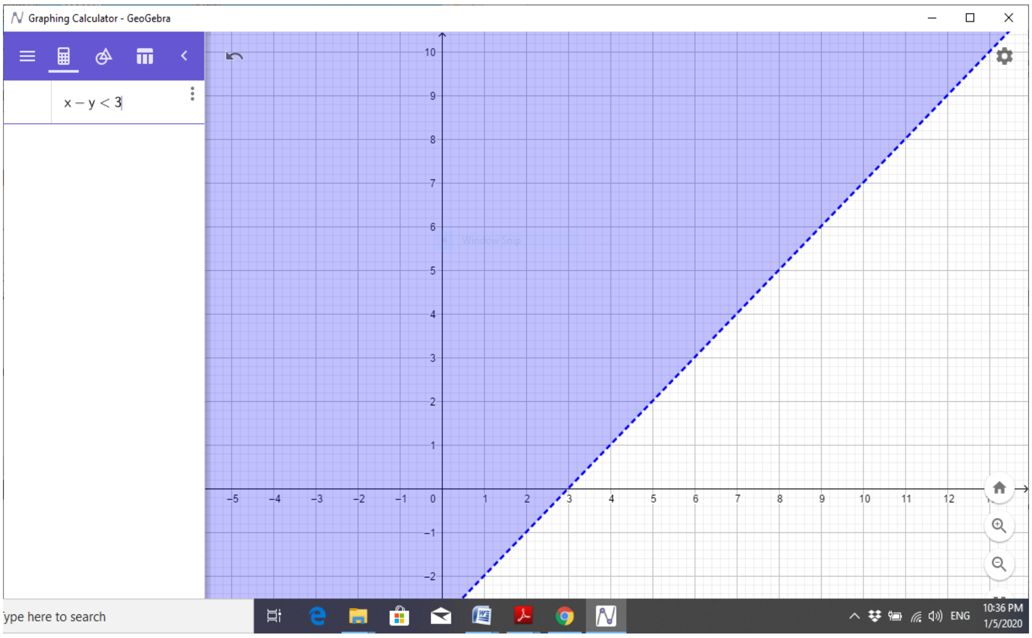
c. Repeat a) and b) for the second inequality. The overlapping area gives the solution of the inequalities.

Worked Example
Shade the region that satisfies the following simultaneous inequalities

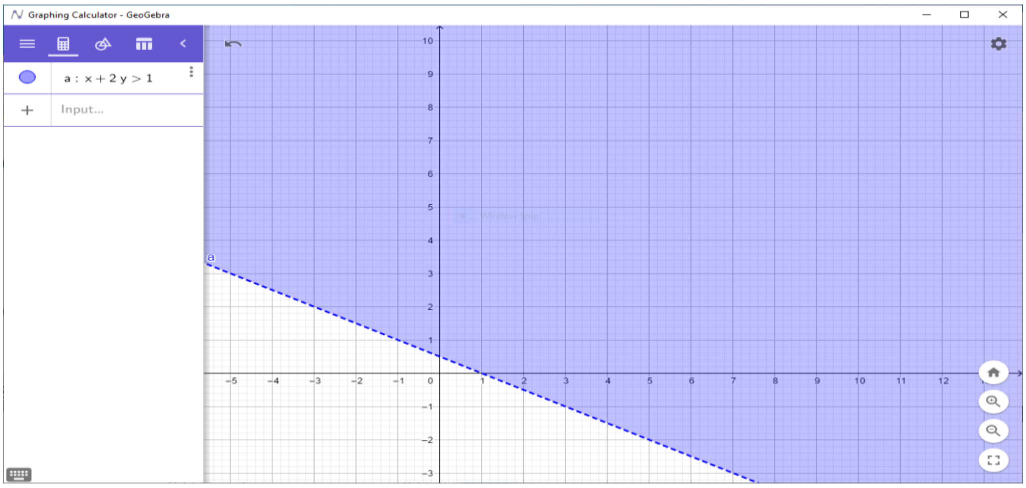
$$x-y<3$$

$$x+2y>3$$

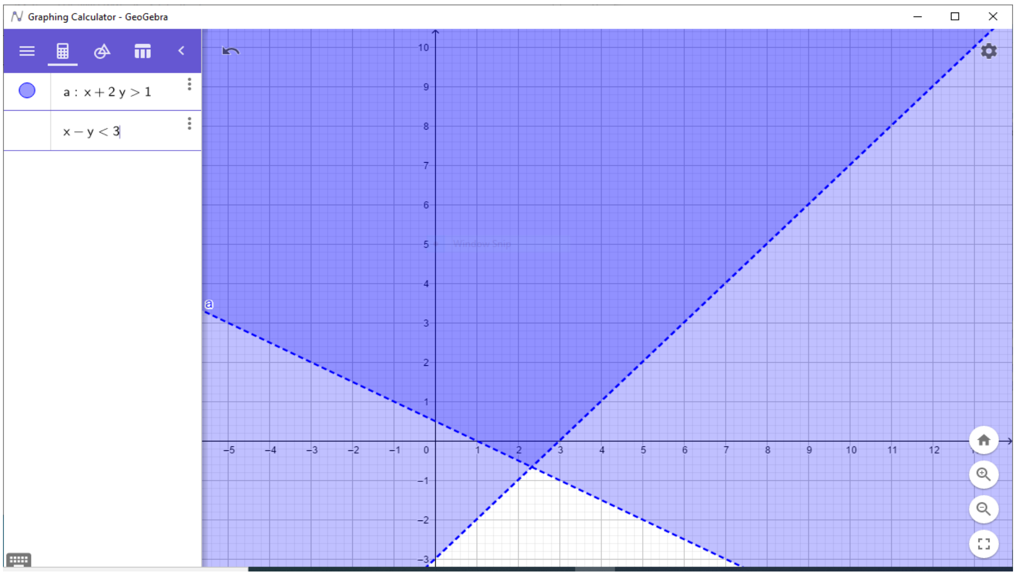
Sketch the solution for each inequality. We start with $x-y<3$.



Now, we sketch the solution for $x+2y>3$ shown below.



Now, the solution is the overlap between the two shaded regions, the area above the x-axis (darker blue), as shown below.

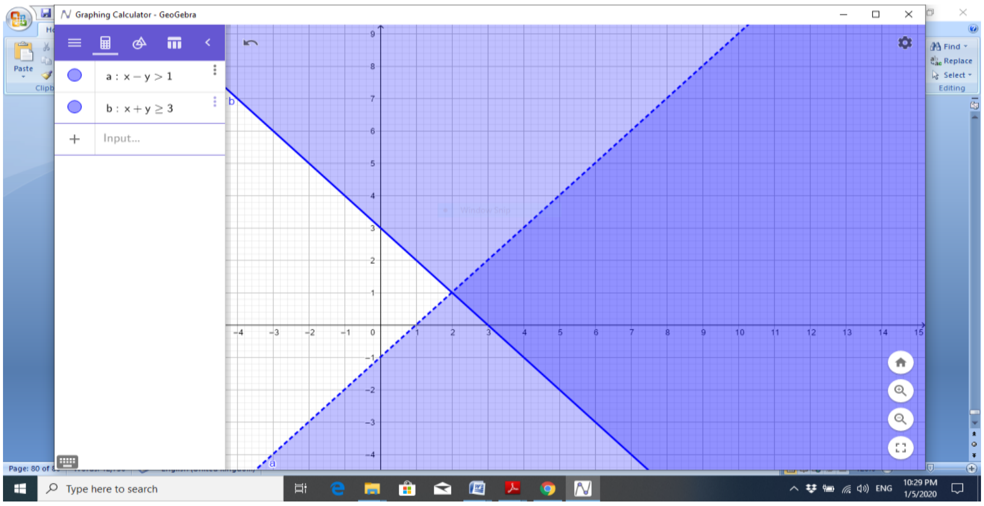


Worked Example

Shade the region that satisfies the following simultaneous inequalities

$$x-y<1$$

$$x+y\geq 3$$



The solution to the systems of inequalities is the region to the right side (darker blue).

Reflection

Shade the region that satisfies each of the following simultaneous inequalities:

a. $x-y<3$
$x+2y>3$

b. $2x-y\leq 6$
$x\geq 3$

c. $x-y\geq 4$
$x+y<3$